Why Wirtinger derivatives behave so well

Bart Michels*

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*Université Paris 13, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse, France. Email: michels@math.univ-paris13.fr

1 Introduction

Given a smooth function $f : \mathbb{C} \to \mathbb{C}$, it makes a priori no sense to talk about the Wirtinger derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$. We could define them by

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\ \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \end{aligned} \tag{1}$$

but that does not explain why we denote them like partial derivatives, where the confusing sign convention comes from, why they behave like partial derivatives, and why they act on power series in z and $\bar{z}$ in the way they should.

Here follows a construction of Wirtinger derivatives that avoids the explicit formula (1). The reader may be interested in the definitions in section 3, in the sanity check in Proposition 3.2, the chain rule from Proposition 4.3 and the results from section 6.

2 Partial derivatives revisited

It is worthwhile to take a second look at what a partial derivative is. Consider finite-dimensional real vector spaces V and W.[1] They carry a canonical smooth structure. Consider a differentiable function $f : V \to W$. At every point $p \in V$, f has a differential $D_p f : V \to W$, an $\mathbb{R}$-linear map. In order to talk about the partial derivatives of f, we need to fix a basis of V, say $(e_1, \ldots, e_n)$, which has a dual basis, $(e_1^*, \ldots, e_n^*)$. Here e_i^* is characterized by the property that $e_i^*(e_j) = \delta_{i,j}$. We can then define

$$\frac{\partial f}{\partial e_i^*}(p) := (D_p f)(e_i) \tag{2}$$

Usually, we take $V = \mathbb{R}^n$ and let (e_i) be the standard basis, whose dual we are used to denote by (x_i). The same holds for a holomorphic map between complex vector spaces, where the differential $D_p f$ is then $\mathbb{C}$-linear.

We see that a partial derivative really is a directional derivative, and we need not fix an entire basis to define the derivative in the direction of e_i. However, in some situations, we are given a linear form $e^* \in V^*$. In order

[1] We may and will also take W to be a complex vector space and regard it as a real one, by restriction of scalars.

to define the derivative w.r.t. e^*, we have to extend it to a basis of V^*, in order to make sense of the antidual vector $e \in V$. We recall:

Proposition 2.1. *Let k be a field and V a finite dimensional k-vector space. Then for every basis (x_i) of V^* there is a unique basis (e_i) of V such that $(x_i) = (e_i^*)$.*

We denote the unique antidual basis of (x_i) by (x_i°). When $k \in \{\mathbb{R}, \mathbb{C}\}$ and (x_i) is a basis of V^*, we thus have, for a linear map $A : V \to W$,

$$\frac{\partial A}{\partial x_i} = A(x_i^\circ) \tag{3}$$

Note that the choice of a basis of V^* is implicit in the LHS.

Proposition 2.2. *Let (x_i) be a basis of V^* and $f : V \to W$ an isomorphism. Then*

$$f(x_i^\circ) = (x_i \circ f^{-1})^\circ$$

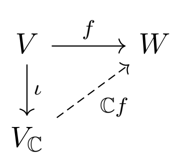
Proof. It suffices to remark that

$$(x_j \circ f^{-1})(f(x_i^\circ)) = \delta_{i,j}$$ □

3 Complexification

Now let W be a complex vector space and consider a differentiable function $f : \mathbb{R}^2 \to W$. Take $p \in \mathbb{R}^2$. Then f has a differential $D_p f : \mathbb{R}^2 \to \mathbb{C}$, which is $\mathbb{R}$-linear. We would like to make sense of $(\partial f/\partial z)(p)$, so we want to view z and $\bar{z}$ as linear forms on $\mathbb{R}^2$ that form a basis of its dual... The solution is to complexify $\mathbb{R}^2$.

Definition 3.1. Let V be a real vector space. Then there exists a complex vector space $V_{\mathbb{C}}$ and an $\mathbb{R}$-linear map $\iota : V \to V_{\mathbb{C}}$ such that for every $\mathbb{R}$-linear map $f : V \to W$ to a complex vector space, there exists a unique $\mathbb{C}$-linear map $\mathbb{C}f : V_{\mathbb{C}} \to W$ such that $f = (\mathbb{C}f) \circ \iota$:



We call $(V_{\mathbb{C}}, \iota)$ a complexification of V.

This is a special case of the adjunction between the extension of scalars and restriction of scalars functors. We may indeed take $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. We call $V_{\mathbb{C}}$ the complexification of V. This gives a functor: for real vector spaces V_1, V_2 and a linear map $f : V_1 \to V_2$, there is a unique $\mathbb{C}$-linear map $f_{\mathbb{C}} : (V_1)_{\mathbb{C}} \to (V_2)_{\mathbb{C}}$ such that the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad f \quad} & V_2 \\ \big\downarrow{\scriptstyle \iota_1} & & \big\downarrow{\scriptstyle \iota_2} \\ (V_1)_{\mathbb{C}} & \overset{f_{\mathbb{C}}}{\dashrightarrow} & (V_2)_{\mathbb{C}} \end{array}$$

In particular, a complexification is unique up to unique $\mathbb{C}$-isomorphism that is compatible with the maps ι.
If we denote $\iota_1 : \mathbb{R} \to \mathbb{C}$ the inclusion, then we have $(\mathbb{R}^n)_{\mathbb{C}} = \mathbb{C}^n$ where the canonical map $\iota_n : \mathbb{R}^n \to \mathbb{C}^n$ is $\iota_1 \times \cdots \times \iota_1$.

3.1 Complexified differentials: two variables

We have the $\mathbb{R}$-linear map $D_p f : \mathbb{R}^2 \to \mathbb{C}$. Passing to the complexification of $\mathbb{R}^2$, this gives us a canonical $\mathbb{C}$-linear map $\mathbb{C}D_p f : \mathbb{C}^2 \to \mathbb{C}$. We want to apply this map to an appropriate vector, and call the result $(\partial f/\partial z)(p)$. To do that, we introduce coordinates on $\mathbb{C}^2$.
We have $\mathbb{R}$-linear maps $z, \bar{z} : \mathbb{R}^2 \to \mathbb{C}$. They give canonical $\mathbb{C}$-linear maps $\mathbb{C}z, \mathbb{C}\bar{z} : \mathbb{C}^2 \to \mathbb{C}$, which are given in coordinates by

$$(\mathbb{C}z)(z_1, z_2) = z_1 + iz_2 \quad , \quad (\mathbb{C}\bar{z})(z_1, z_2) = z_1 - iz_2$$

They form a basis of the dual $(\mathbb{C}^2)^*$, and we define, in analogy to (3):

$$\begin{aligned} \frac{\partial f}{\partial z}(p) &:= (\mathbb{C}D_p f)((\mathbb{C}z)^\circ) \\ \frac{\partial f}{\partial \bar{z}}(p) &:= (\mathbb{C}D_p f)((\mathbb{C}\bar{z})^\circ) \end{aligned} \tag{4}$$

It is a straightforward calculation that these definitions coincide with (1). Explicitly, $(\mathbb{C}z)^\circ = (\frac{1}{2}, -\frac{i}{2})^t$ and $(\mathbb{C}\bar{z})^\circ = (\frac{1}{2}, \frac{i}{2})^t$. But we don't like to compute, so we will instead derive those formulas from the chain rule from Proposition 4.2.

3.2 Complexified differentials: multiple variables

Take $n \geqslant 1$ and let $f : \mathbb{C}^n \to \mathbb{C}$ be differentiable. We denote the coordinates on $\mathbb{C}^n$ by (z_i). We want to understand what $\partial f/\partial z_i$ means.

Take $p \in \mathbb{C}^n$. As in the two variable case, we have a $\mathbb{C}$-linear map $\mathbb{C}D_pf : \mathbb{C}^{2n} \to \mathbb{C}$, and we have coordinates $\mathbb{C}z_i, \mathbb{C}\bar{z}_i$ on $\mathbb{C}^{2n}$. We define

$$\begin{aligned} \frac{\partial f}{\partial z_i}(p) &:= (\mathbb{C}D_pf)((\mathbb{C}z_i)^\circ) \\ \frac{\partial f}{\partial \bar{z}_i}(p) &:= (\mathbb{C}D_pf)((\mathbb{C}\bar{z}_i)^\circ) \end{aligned} \tag{5}$$

Using the universal property of complexification, one verifies that these definitions do not depend on the choice of complexification of $\mathbb{C}^n$.
So far this was just language. What we really want is to answer the questions from the introduction. For now, we remark the following:

Proposition 3.2. *Write the standard complex coordinates on $\mathbb{C}^n$ as (z_i). Then*

$$\begin{aligned} \frac{\partial z_i}{\partial z_j} = \frac{\partial \bar{z}_i}{\partial \bar{z}_j} = \delta_{i,j} \\ \frac{\partial z_i}{\partial \bar{z}_j} = \frac{\partial \bar{z}_i}{\partial z_j} = 0 \end{aligned}$$

Proof. The map $z_i : \mathbb{C}^n \to \mathbb{C}$ is $\mathbb{R}$-linear, so its differential equals itself: $D_p(z_i) = z_i$. Thus the extended differential $\mathbb{C}D_p(z_i)$ equals the coordinate $\mathbb{C}z_i$ of $\mathbb{C}^{2n}$. Similarly for $\bar{z}_i$. The statement now follows from the definition of $\partial/\partial z_i$ and $\partial/\partial \bar{z}_i$, and the definition of the antidual basis. □

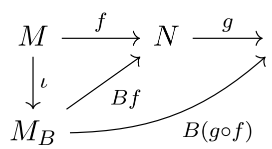
Defining the partial derivatives in terms of the antidual vectors $(\mathbb{C}z_i)^\circ, (\mathbb{C}\bar{z}_i)^\circ$ has a great advantage. We need not know exactly what those vectors are, yet we know the coordinates of each $w \in \mathbb{C}^{2n}$ in this basis: they are $z_i(w), \bar{z}_i(w)$. This observation will serve us when we prove chain rules, in the next section.

3.3 Conjugation

The end goal of this subsection is Proposition 3.5. The remainder of the text does not rely on it, so it may safely be skipped.
For every $\mathbb{R}$-automorphism σ of $\mathbb{C}$, a complexification $V_\mathbb{C}$ has a canonical $\mathbb{R}$-automorphism given by $1 \otimes \sigma : V \otimes_\mathbb{R} \mathbb{C} \to V \otimes_\mathbb{R} \mathbb{C}$, conjugated by the unique $\mathbb{C}$-isomorphism $V_\mathbb{C} \cong V \otimes_\mathbb{R} \mathbb{C}$ compatible with ι. This allows us to define conjugation on $V_\mathbb{C}$. The conjugation may also be constructed intrinsically as follows. Define another $\mathbb{C}$-linear structure on $V_\mathbb{C}$ by precomposing the ring representation $\mathbb{C} \to \mathrm{End}_\mathbb{Z}(V_\mathbb{C})$ with complex conjugation. Call the resulting $\mathbb{C}$-vector space $V'_\mathbb{C}$. We can define conjugation as the unique $\mathbb{C}$-homomorphism $V_\mathbb{C} \to V'_\mathbb{C}$ compatible with ι, given by the universal property

of $V_{\mathbb{C}}$. It is not hard to see that, because conjugation is an automorphism of $\mathbb{C}$, we have that $(V'_{\mathbb{C}}, \iota)$ is still a complexification of V, which implies that the conjugation on $V_{\mathbb{C}}$ is an automorphism.
More generally, given commutative rings A, B, a ring homomorphism $g : A \to B$, an A-endomorphism σ of B and an A-module M, the endomorphism σ induces a canonical A-linear M-endomorphism of the extension of scalars $M_B = g_* M$. It can be constructed in two ways, as before. We denote it by $M \otimes \sigma$, because we may also view it as the image of σ under the functor $M \otimes -$.
The map $f = M \otimes \sigma$ is σ-semilinear: it satisfies, by construction, $f(lm) = \sigma(l)f(m)$ for $m \in M_B$, $l \in L$. If M, N, P are B-modules, $f : M \to N$ is a σ-semilinear and $g : N \to P$ is τ-semilinear, then $g \circ f$ is $(\tau \circ \sigma)$-semilinear.

Proposition 3.3. *Let M be an A-module, (M_B, ι) its extension of scalars, N, P be B-modules, $f : M \to N$ be A-linear, $\sigma \in \mathrm{Aut}_A(B)$, $g : N \to P$ be σ-semilinear. Take Bf and $B(g \circ f)$ as in the universal property of M_B:*



Then

$$B(g \circ f) = g \circ Bf \circ (M \otimes \sigma^{-1})$$

Proof. The RHS is a B-linear map extending $g \circ f$. □

Proposition 3.4. *Let K be a field, L/K a field extension, $\sigma \in \mathrm{Aut}_K(L)$, (x_i) a basis of V_L^* and $f : V \to W$ an σ-semilinear isomorphism. (That is, a σ-semilinear bijection.) Then*

$$f(x_i^\circ) = (\sigma \circ x_i \circ f^{-1})^\circ$$

Proof. Note that $\sigma \circ x_i \circ f^{-1}$ is indeed L-linear. It thus suffices to remark that

$$(\sigma \circ x_j \circ f^{-1})(f(x_i^\circ)) = \sigma(\delta_{i,j}) = \delta_{i,j}$$ □

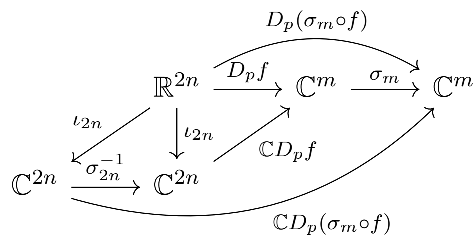
Because $\mathbb{C}^n$ is the complexification of $\mathbb{R}^n$, it has a canonical conjugation map, denoted σ_n. Explicitly, it is given by $(z_i)^t \mapsto (\bar{z}_i)^t$.

Proposition 3.5 (Conjugation distributes over fractions)**.** *Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be differentiable. Then*

$$\sigma_m \circ \frac{\partial f}{\partial z_i} = \frac{\partial(\sigma_m \circ f)}{\partial(\sigma_1 \circ z_i)}$$

(Here, $\sigma_1 \circ z_i = \bar{z}_i$.)

Proof. We have a commutative diagram



where $\mathbb{C}D_p(\sigma_m \circ f) = \sigma_m \circ \mathbb{C}D_p f \circ \sigma_{2n}^{-1}$ by Proposition 3.3.[2] We now look at the image of $(\mathbb{C}\bar{z}_i)^\circ$ under this map. On the one hand, it equals $\partial(\sigma_m \circ f)/\partial \bar{z}_i$. On the other hand,

$$\begin{aligned}(\sigma_m \circ \mathbb{C}D_p f \circ \sigma_{2n}^{-1})((\mathbb{C}\bar{z}_i)^\circ) &= (\sigma_m \circ \mathbb{C}D_p f)((\sigma_1^{-1} \circ \mathbb{C}\bar{z}_i \circ \sigma_{2n})^\circ) \\ &= (\sigma_m \circ \mathbb{C}D_p f)((\mathbb{C}(\sigma_1^{-1} \circ \bar{z}_i))^\circ) \\ &= (\sigma_m \circ \mathbb{C}D_p f)((\mathbb{C}z_i)^\circ) \\ &= \sigma_m \left(\frac{\partial f}{\partial z_i}(p) \right)\end{aligned}$$

where we used Propositions 3.4 and 3.3. □

Remark 3.6. The chain rule (Proposition 4.3) does not allow to directly prove Proposition 3.5.

4 Chain rule

For differentiable functions $f : \mathbb{R}^a \to \mathbb{R}^b$ and $g : \mathbb{R}^b \to \mathbb{R}^c$, the chain rule

$$\frac{\partial(f \circ g)}{\partial x_i} = \sum_{j=1}^{b} \frac{\partial g}{\partial x_j} \circ f_j \cdot \frac{\partial f_j}{\partial x_i}$$

is the coordinate version of the identity

$$D_p(g \circ f) = D_{f(p)}(g) \circ D_p(f)$$

For practical purposes, we want to study chain rules involving the Wirtinger derivatives $\partial/\partial z_i, \partial/\partial \bar{z}_i$. We place ourselves immediately in the setting of

[2]Of course, we have $\sigma_{2n}^{-1} = \sigma_{2n}$, but we are really using that the semilinearity of σ_m cancels with that of σ_{2n}^{-1}.

multiple variables. Consider differentiable functions $f : \mathbb{C}^a \to \mathbb{C}^b$ and $g : \mathbb{C}^b \to \mathbb{C}^c$.[3]

Denote the complex coordinates on $\mathbb{C}^n$ by (z_i); the real coordinates by (x_i, y_i), so that $x_i = \Re\mathrm{e} \circ z_i$ and $y_i = \Im\mathrm{m} \circ z_i$.

Proposition 4.1. *We have*

$$\frac{\partial(g \circ f)}{\partial x_i} = \sum_{j=1}^{b} \left(\frac{\partial g}{\partial z_j} \circ f \cdot \frac{\partial f_j}{\partial x_i} + \frac{\partial g}{\partial \bar{z}_j} \circ f \cdot \frac{\partial \bar{f}_j}{\partial x_i} \right)$$

$$\frac{\partial(g \circ f)}{\partial y_i} = \sum_{j=1}^{b} \left(\frac{\partial g}{\partial z_j} \circ f \cdot \frac{\partial f_j}{\partial x_i} + \frac{\partial g}{\partial \bar{z}_j} \circ f \cdot \frac{\partial \bar{f}_j}{\partial y_i} \right)$$

Note that $\partial \bar{f}_j / \partial x_i = \overline{\partial f_j / \partial x_i}$, because (by the chain rule for differentiable functions) $\partial / \partial x_i$ commutes with $\mathbb{R}$-linear maps, in particular, with complex conjugation.

Proof. We want a coordinate-free version of the chain rule in the statement. We have a commutative diagram

$$\begin{array}{ccccc} \mathbb{C}^a & \xrightarrow{D_p f} & \mathbb{R}^{2b} & \xrightarrow{D_{f(p)} g} & \mathbb{C}^c \\ & & \downarrow \iota_{2b} & \nearrow_{\mathbb{C}D_{f(p)}g} & \\ & & \mathbb{C}^{2b} & & \end{array}$$

The chain rule $D_p(g \circ f) = D_{f(p)} g \circ D_p f$ implies

$$D_p(g \circ f) = (\mathbb{C}D_{f(p)} g) \circ \iota_{2b} \circ D_p f$$

which is the statement of the proposition, when we choose the coordinates (x_i, y_i) on $\mathbb{C}^a$ and $(\mathbb{C}z_i, \mathbb{C}\bar{z}_i)$ on $\mathbb{C}^{2b}$. □

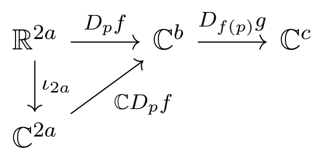
Proposition 4.2. *We have*

$$\frac{\partial(g \circ f)}{\partial z_i} = \sum_{j=1}^{b} \left(\frac{\partial g}{\partial x_j} \circ f \cdot \frac{\partial \Re\mathrm{e}\, f_j}{\partial z_i} + \frac{\partial g}{\partial y_j} \circ f \cdot \frac{\partial \Im\mathrm{m}\, f_j}{\partial z_i} \right)$$

$$\frac{\partial(g \circ f)}{\partial \bar{z}_i} = \sum_{j=1}^{b} \left(\frac{\partial g}{\partial x_j} \circ f \cdot \frac{\partial \Re\mathrm{e}\, f_j}{\partial \bar{z}_i} + \frac{\partial g}{\partial y_j} \circ f \cdot \frac{\partial \Im\mathrm{m}\, f_j}{\partial \bar{z}_i} \right)$$

[3] Restricting ourselves to the case where $a = 1$ or $c = 1$ would mean that we are already assuming a chain rule in certain cases (where g is a projection map) or that $\partial f / \partial z_i = \partial f(z_1, \ldots, z_a) / \partial z_i$ for fixed $(z_j)_{j \neq i}$ (also a particular case of the chain rule).

Proof. We have a commutative diagram



The claim follows from

$$D_p(g \circ f) = D_{f(p)} g \circ (\mathbb{C}D_{f(p)} g) \circ \iota_{2a}$$

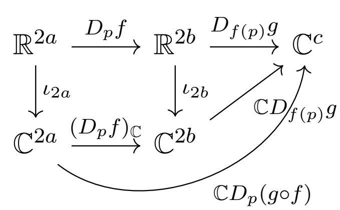
when we choose the coordinates $(\mathbb{C}z_i, \mathbb{C}\bar{z}_i)$ on $\mathbb{C}^{2a}$ and (x_i, y_i) on $\mathbb{C}^b$. $\square$

In particular, the second chain rule implies, together with Proposition 3.2, the usual formula (1) for the Wirtinger derivatives.

Proposition 4.3. *We have*

$$\frac{\partial(g \circ f)}{\partial z_i} = \sum_{j=1}^{b} \left(\frac{\partial g}{\partial z_j} \circ f \cdot \frac{\partial f_j}{\partial z_i} + \frac{\partial g}{\partial \bar{z}_j} \circ f \cdot \frac{\partial \bar{f}_j}{\partial z_i} \right)$$
$$\frac{\partial(g \circ f)}{\partial \bar{z}_i} = \sum_{j=1}^{b} \left(\frac{\partial g}{\partial z_j} \circ f \cdot \frac{\partial f_j}{\partial \bar{z}_i} + \frac{\partial g}{\partial \bar{z}_j} \circ f \cdot \frac{\partial \bar{f}_j}{\partial \bar{z}_i} \right)$$

Proof. If we wanted to use explicit formulas for Wirtinger derivatives, this follows from the first or the second chain rule by linearity. But we don't want to do that, so we give a direct proof via a coordinate-free chain rule. We have a commutative diagram



The fact that

$$\mathbb{C}D_p(g \circ f) = (\mathbb{C}D_{f(p)} g) \circ (D_p f)_{\mathbb{C}}$$

follows from the uniqueness in the definition of the complexification of $\mathbb{R}^{2a}$: both sides are extensions of $D_p(g \circ f)$ to $\mathbb{C}^{2a}$. When we choose coordinates $(\mathbb{C}z_i, \mathbb{C}\bar{z}_i)$ on $\mathbb{C}^{2a}$ and $\mathbb{C}^{2b}$, that equality is what we wanted to prove. $\square$

5 Holomorphic functions

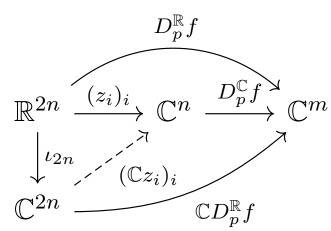
Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic function. It is in particular differentiable, so viewing it as a function $\mathbb{R}^{2n} \to \mathbb{R}^{2m}$ gives us Wirtinger derivatives $\partial f/\partial z_i, \partial f/\partial \bar{z}_i$. On the other hand, there are the complex partial derivatives defined by

$$\frac{\partial f}{\partial_{\mathbb{C}} z_i}(p) = (D_p^{\mathbb{C}} f)(z_i^{\circ})$$

Proposition 5.1. *For a holomorphic $f : \mathbb{C}^n \to \mathbb{C}^m$, we have*

$$\frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial_{\mathbb{C}} z_i}$$
$$\frac{\partial f}{\partial \bar{z}_i} = 0$$

Proof. Because f is holomorphic, at every point p it has a best $\mathbb{C}$-linear approximation $D_p^{\mathbb{C}} f$. Because the $\mathbb{R}$-isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$ respects the norm, it is in particular a best $\mathbb{R}$-linear approximation. Thus, under the identification $(z_i)_i : \mathbb{R}^{2n} \xrightarrow{\sim} \mathbb{C}^n$, the differentials $D_p^{\mathbb{R}} f$ and $D_p^{\mathbb{C}} f$ coincide. Because $D_p^{\mathbb{C}} f$ is $\mathbb{C}$-linear, the unique map $\mathbb{C}D_p^{\mathbb{R}} f$ given by the universal property of complexification, factors through $\mathbb{C}^n$:



We can now compute

$$\begin{aligned}(\mathbb{C}D_p^{\mathbb{R}} f)((\mathbb{C}z_i)^{\circ}) &= (D_p^{\mathbb{C}} f)\,[(\mathbb{C}z_i)_i((\mathbb{C}z_i)^{\circ})] \\ &= (D_p^{\mathbb{C}} f)(e_i) \\ &= (D_p^{\mathbb{C}} f)(z_i^{\circ})\end{aligned}$$

and

$$\begin{aligned}(\mathbb{C}D_p^{\mathbb{R}} f)((\mathbb{C}\bar{z}_i)^{\circ}) &= (D_p^{\mathbb{C}} f)\,[(\mathbb{C}z_i)_i((\mathbb{C}\bar{z}_i)^{\circ})] \\ &= (D_p^{\mathbb{C}} f)(0) \\ &= 0\end{aligned}$$

□

6 Power series

Define smooth maps $\Delta_n : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ by

$$\Delta_n(z_i) := ((z_i), (\bar{z}_i))$$

Proposition 6.1 (Diagonal restriction)**.** *Let $f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^c$ be differentiable. Write the standard coordinates on $\mathbb{C}^n \times \mathbb{C}^n$ as (z_i, w_i). Then*

$$\frac{\partial(f \circ \Delta_n)}{\partial z_i} = \frac{\partial f}{\partial z_i} \circ \Delta_n$$
$$\frac{\partial(f \circ \Delta_n)}{\partial \bar{z}_i} = \frac{\partial f}{\partial w_i} \circ \Delta_n$$

Proof. This follows from the third chain rule. By Proposition 3.2, only one term survives. ☐

Proposition 6.2. *Let $(a_{k,l})_{k,l \in \mathbb{N}^n} \in \mathbb{C}$ and suppose that the series*

$$f(z) = \sum_{k,l \in \mathbb{N}^n} a_{k,l} z^k \bar{z}^l$$

converges for z in some open polydisk $D \subset \mathbb{C}^n$ centered at 0. Then for z in that polydisk,

$$\frac{\partial f}{\partial z_i} = \sum_{k,l \in \mathbb{N}^n} a_{k,l} k_i z^{k-e_i} \bar{z}^l$$
$$\frac{\partial f}{\partial \bar{z}_i} = \sum_{k,l \in \mathbb{N}^n} a_{k,l} l_i z^k \bar{z}^{l-e_i}$$

Proof. The convergence of the series in the polydisk D implies a bound on the coefficients, which in its turn implies that the series

$$g(z, w) = \sum_{k,l \in \mathbb{N}^n} a_{k,l} z^k w^l$$

converges in the polydisk $D \times D \subset \mathbb{C}^n \times \mathbb{C}^n$. Using the previous proposition, the claim reduces to a standard fact about holomorphic power series. ☐

7 Concluding remarks

At some point, one has to acknowledge that the Wirtinger derivatives are complex linear combinations of the $\partial/\partial x_i$ and $\partial/\partial y_i$. For example, if one wants to show that for $f : \mathbb{C} \to \mathbb{C}$ of class C^2, the Wirtinger derivatives commute with each other, or that $(\partial f/\partial z)(p)$ depends only on the values of f in a neighborhood of p. But hopefully the reader will be convinced that some of the more algebraic properties of Wirtinger derivatives can be explained conceptually.

We haven't been very careful about the domain of definition of our differentiable functions, assuming most of the time that they are defined on all of $\mathbb{C}^n$. The extension to functions defined on open subsets, as well as differentiable functions between manifolds with their partial derivatives with respect to a chart, is straightforward: the differential $D_p f$ becomes a linear map between real tangent spaces, which we can complexify. The complexification is less explicit, but we have taken care to state everything as canonical as possible, so that the extra work should be minimal.