# Riemannian distances are locally equivalent

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## Contents



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All manifolds and submanifolds are assumed to be smooth and connected.

# 1 Equivalence of norms

**Proposition 1.1.** *Let $V$ be an $n$-dimensional real vector space with norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Let $(e_i)$ be a basis of $V$. Then*

$$\|v\|_1 \le \frac{n \cdot \max_i \|e_i\|_1}{\min_{\|w\|_\infty = 1} \|w\|_2} \cdot \|v\|_2$$

*for all $v \in V$, where the sup-norm is w.r.t. the basis $(e_i)$.*

*Proof.* We have

$$\left\|\sum a_i e_i\right\|_1 \le n \cdot \max_i \|e_i\|_1 \cdot \left\|\sum a_i e_i\right\|_\infty$$

and

$$\begin{aligned} \|v\|_2 &= \left\| \frac{v}{\|v\|_\infty} \right\|_2 \cdot \|v\|_\infty \\ &\ge \min_{\|w\|_\infty = 1} \|w\|_2 \cdot \|v\|_\infty \end{aligned}$$

where $\min_{\|w\|_\infty=1} \|w\|_2 > 0$ because the unit sphere for the norm $\|\cdot\|_\infty$ is compact (closed and bounded). □

**Proposition 1.2.** *Let $M$ be a smooth manifold of dimension $n$ and $g_1, g_2$ Riemannian metrics on $M$. Let $K \subset M$ be compact. Then there exist $c_1, c_2 > 0$ such that for all $x \in K$ and $v \in T_xM$ we have*

$$c_1\|v\|_2 \le \|v\|_1 \le c_2\|v\|_2 \,.$$

*Proof.* It suffices to prove that every $x \in M$ has a neighborhood $U$ for which there exist $c_1, c_2 > 0$ such that the above inequalities hold for all $y \in U$, $v \in T_yM$. Fix $x \in M$ and let $U_0$ be a coordinate neighborhood of $x$, so that there exists a chart $\phi : U_0 \to V \subset \mathbb{R}^n$ and the partial derivatives $(\partial/\partial\phi_i)|_y$ form a smooth basis of $T_yM$ for $y \in U_0$. Define the sup-norm $\|\cdot\|_\infty$ in $T_yM$ w.r.t. this basis. Let $U \subset U_0$ be relatively compact in $U_0$. Then by Proposition 1.1,

$$\|v\|_1 \le \frac{n \cdot \max_i \|(\partial/\partial\phi_i)|_y\|_1}{\min_{\|w\|_\infty=1} \|w\|_2} \|v\|_2$$

for $y \in U_0$, $v \in T_yM$, so we can take

$$c_2 = \frac{n \cdot \sup_{y\in U} \max_i \|(\partial/\partial\phi_i)|_y\|_1}{\inf_{y \in U} \min_{\|w\|_\infty=1} \|w\|_2} \,.$$

The other inequality follows by symmetry. □

## 2 The radius of compactness

This section introduces the notion of *radius of compactness*. This is not strictly necessary for the sequel, but it is a neat notion with some nice properties, and gives an excuse to review some standard facts from topology that will be used without reference in the next sections. When $M$ is a metric space and $x \in M$, denote

$$\overline{B}(x, R) = \{y \in M : d(x, y) \leq R\}$$

for the closed ball of radius $R$ centered at $x$.

**Definition 2.1.** *Let $M$ be a metric space and $x \in M$. Define the radius of compactness of $x$ by*

$$\mathrm{RC}(x) := \sup\{R \geq 0 : \overline{B}(x, R) \text{ is compact}\} \in [0, \infty]\,.$$

**Proposition 2.2.** *Let $M$ be a metric space.*

1. *Either* $\mathrm{RC}(x) = \infty$ *for all* $x \in M$, *or* $|\mathrm{RC}(y) - \mathrm{RC}(x)| \leq d(x, y)$ *for all* $x, y \in M$.

2. $\mathrm{RC}$ *is continuous for the order topology on* $[0, \infty]$.

*Proof.* 1. We show that $\mathrm{RC}(y) \leq \mathrm{RC}(x) + d(x, y)$ for all $x, y \in M$, from which both statements follow. (Here $\infty + a = \infty$ for all $a \in [0, \infty]$.) Suppose $\mathrm{RC}(y) > \mathrm{RC}(x) + d(x, y)$. Then there exists a closed compact ball of radius strictly larger than $\mathrm{RC}(x) + d(x, y)$ centered at $y$. But this ball contains a ball of radius strictly larger than $\mathrm{RC}(x)$ centered at $x$. Contradiction.

2. It is either constant and equal to $\infty$, or real-valued and Lipschitz, hence continuous. □

**Remark 2.3.** A metric space is called *proper* or *ball-compact* when $\mathrm{RC}(x) = \infty$ for one (hence every) $x \in M$. Equivalently, when its closed and bounded subsets are compact.

**Proposition 2.4.** *Let $M$ be a metric space and $K \subset M$ compact.*

1. *If $M$ is locally compact, there exists $R > 0$ such that $\overline{B}(x, R)$ is compact for all $x \in K$.*

2. *If $R$ is as above and $r < R$, then*

$$\overline{B}(K, r) := \bigcup_{x \in K} \overline{B}(x, r)$$

*is compact.*

*Proof.* 1. Because $K$ is compact, so is $\mathrm{RC}(K)$. Because $M$ is locally compact, $0 \notin \mathrm{RC}(K)$. Because $\mathrm{RC}(K)$ is closed, there exists $R > 0$ such that $\mathrm{RC}(x) > R$ for all $x \in K$.

2. We show that $\overline{B}(K,r)$ is complete and totally bounded. Complete: Let $(x_n)$ be a Cauchy sequence in $\overline{B}(K,r)$. There exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \leq R - r$ for all $m, n \geq N$. If $x \in K$ is such that $x_N \in \overline{B}(x,r)$, then the sequence $(x_n)$ is eventually contained in the compact set $\overline{B}(x,R)$. In particular, $(x_n)$ converges in $\overline{B}(x,R)$. Because $\overline{B}(K,r)$ is closed in $M$, the limit point lies in $\overline{B}(K,r)$.

   Totally bounded: We can cover the compact set $K$ by finitely many open balls of radius $R-r$, say centered at $x_1, \ldots, x_N$. Let $\epsilon > 0$. We can cover each of the compact balls $\overline{B}(x_i, R)$ by finitely many balls of radius $\epsilon$. We have then covered all the balls $\overline{B}(x,r)$ for $x \in K$. Thus $\overline{B}(K,r)$ is totally bounded. □

This leads us to define:

**Definition 2.5.** *Let $M$ be a metric space and $K \subset M$ compact. Define the radius of compactness*

$$\mathrm{RC}(K) := \sup\{R \geq 0 : \overline{B}(K,R) \textit{ is compact}\} \in [0,\infty] .$$

From Proposition 2.4 it follows that

$$\mathrm{RC}(K) = \min_{x \in K} \mathrm{RC}(x) .$$

In general, the set $\overline{B}(K, \mathrm{RC}(K))$ may or may not be compact:

**Example 2.6.** 1. Let $M \subset \mathbb{R}$ be an open interval and $x \in M$. Then $\overline{B}(x, \mathrm{RC}(x))$ is not compact.

2. Let $M \subset \mathbb{R}^2$ consist of an open ball and one isolated point $x$. Then $\overline{B}(x, \mathrm{RC}(x)) = \{x\}$ is compact.

3. Let $M \subset \mathbb{R}^2$ consist of the points $(1/n, 0)$ and $(1/n, 1)$ for $n \in \mathbb{N}_{>0}$, and the point $(0,0)$. Let $K = M \cap \mathbb{R} \times \{0\}$. Then $\mathrm{RC}(0,0) = 1$ and $\overline{B}(x,1)$ is compact for all $x \in K$, so that $\mathrm{RC}(K) = 1$. We have that $\overline{B}(K,1)$ is totally bounded but not complete.

4. Let $M \subset \mathbb{R} \times \ell_\infty(\mathbb{R})$ (with the sup distance) consist of the points $(1/n, 0)$ and $(1/n, e_n)$ for $n \in \mathbb{N}_{>0}$, and the point $(0,0)$. Here, $e_n$ is the sequence with a 1 at position $n$ and 0 elsewhere. Let $K = M \cap (\mathbb{R} \times \{0\})$. Then $\mathrm{RC}(0,0) = 1$ and $\overline{B}(x,1)$ is compact for all $x \in K$, so that $\mathrm{RC}(K) = 1$. We have that $\overline{B}(K,1)$ is complete but not totally bounded.

5. Take the product of the previous two examples, equipped with the sup-distance. Then $\overline{B}(x,1)$ is compact for all $x \in K$ and $\mathrm{RC}(K) = 1$. We have that $\overline{B}(K,1)$ is neither complete nor totally bounded.

We will not need this, but to be complete, we state a sufficient condition for $\overline{B}(x, \mathrm{RC}(x))$ to be non-compact.

**Proposition 2.7.** *Let $M$ be a locally compact metric space such that for all $x \in M$ and $R, r > 0$ we have*

$$\overline{B}(x, R + r) = \overline{B}(\overline{B}(x, R), r)$$

*(E.g. when $M$ is a complete Riemannian manifold with its Riemannian distance.) Let $x \in M$ and $K \subset M$ compact.*

1. *If $\overline{B}(x, \mathrm{RC}(x))$ is compact, then $\mathrm{RC}(x) = \infty$.*
2. *If $\overline{B}(K, \mathrm{RC}(K))$ is compact, then $\mathrm{RC}(K) = \infty$.*

*Proof.*

1. Suppose $\mathrm{RC}(x) < \infty$ and $\overline{B}(x, \mathrm{RC}(x))$ is compact. By Proposition 2.4, there exists $r > 0$ such that

$$\overline{B}(\overline{B}(x, \mathrm{RC}(x)), r) = \overline{B}(x, \mathrm{RC}(x) + r)$$

is compact. This contradicts the maximality of $\mathrm{RC}(x)$.

2. Because $\mathrm{RC}(K) = \min_{x \in K} \mathrm{RC}(x)$, there exists $x \in K$ such that $\overline{B}(x, \mathrm{RC}(x))$ is compact. By the first part, we have $\mathrm{RC}(K) = \infty$. □

# 3 Equivalence of distances

**Proposition 3.1.** *Let $M$ be a smooth manifold of dimension $n$ and $g_1, g_2$ Riemannian metrics on $M$. Let $K \subset M$ be compact. Then there exist $c_1, c_2 > 0$ such that for all $x_1, x_2 \in K$ we have*

$$c_1 d_2(x_1, x_2) \leq d_1(x_1, x_2) \leq c_2 d_2(x_1, x_2) .$$

*Proof.* We show the right inequality; the left follows by symmetry. By Proposition 2.4, there exists $r > 0$ such that $K' = \overline{B}_2(K, r)$ is compact. First note that for points with $d_2(x_1, x_2) \geq r$ we can take

$$c_2 = \max_{x_1, x_2 \in K} d_1(x_1, x_2)/r .$$

We may thus assume that $d_2(x_1, x_2) \leq r$. Let $0 < \epsilon \leq r$ and $\gamma : [0, 1] \to M$ be a piecewise smooth regular curve joining $x_1$ and $x_2$, with $\mathrm{length}_2(\gamma) \leq d_2(x_1, x_2) + \epsilon \leq 2r$. Then the image of $\gamma$ must lie in $K'$. Let $c_2'$ be the constant from Proposition 1.2, applied to the compact set $K'$. Then

$$\begin{aligned} \mathrm{length}_1(\gamma) &= \int_0^1 \|\gamma'(t)\|_1 dt \\ &\leq c_2' \int_0^1 \|\gamma'(t)\|_2 dt \\ &= c_2' \cdot \mathrm{length}_2(\gamma) , \end{aligned}$$

so that $d_1(x_1, x_2) \leq c_2'(d_2(x_1, x_2) + \epsilon)$. Taking $\epsilon > 0$ arbitrarily small, it follows that $d_1(x_1, x_2) \leq c_2' d_2(x_1, x_2)$. We can thus take

$$c_2 = \max(c_2', \max_{x_1, x_2 \in K} d_1(x_1, x_2)/r) .$$

□

**Proposition 3.2.** *Let $M$ be a smooth manifold of dimension $n$ and $g_1, g_2$ Riemannian metrics on $M$. Let $K \subset M$ be compact and $S \subset M$ nonempty. Then there exist $c_1, c_2 > 0$ such that for all $x \in K$ we have*

$$c_1 d_2(x, S) \le d_1(x, S) \le c_2 d_2(x, S) \,.$$

*Proof.* Again, we limit ourselves to the left inequality. Let $r > 0$ be such that $K' = \overline{B}_2(K, 2r)$ is compact. For points with $d_2(x, S) \ge r$, we can take $c_2 = \max_{x\in K} d_1(x, S)/r$. Take $x \in K$ with $d_2(x, S) \le r$. Let $0 < \epsilon \le r$ and let $s \in S$ such that $d_2(x, s) \le d_2(x, S) + \epsilon \le 2r$. Then $s \in K'$. Let $c_2'$ be the constant from Proposition 3.1 applied to the compact set $K'$. Then

$$d_1(x, S) \le d_1(x, s) \le c_2' d_2(x, s) \le c_2'(d_2(x, S) + \epsilon) \,.$$

Taking $\epsilon > 0$ arbitrarily small, the conclusion follows with

$$c_2 = \max(c_2', \max_{x\in K} d_1(x, S)/r) \,. \qquad \square$$

**Remark 3.3.** The proofs of Proposition 3.1 and 3.2 simplify when we assume that $d_1$ and $d_2$ are complete: by Hopf–Rinow, the compactness of $K$ then implies the compactness of $\overline{B}_2(K, r)$ for all $r \ge 0$.

**Example 3.4.**

1. Let $G$ be a real Lie group with a Riemannian distance and $U \subset \mathfrak{g} = T_e G$ such that $\exp : U \to G$ is a diffeomorphism on its image. Equip $\mathfrak{g}$ with a norm. Then for $K \subset U$ compact and $X, Y \in K$,

$$d_G(e^X, e^Y) \asymp \|X - Y\| \,,$$

   by Proposition 3.1 applied to the metric on $G$ and the pushforward of a Euclidean metric on $\mathfrak{g}$ by exp.

2. Let $M, N$ be smooth manifolds and equip $M, N$ and $M \times N$ with Riemannian metrics $g_M, g_N, g_{M\times N}$. Let $K \subset M \times N$ be compact. Then for $(x_M, x_N), (y_M, y_N) \in K$,

$$d_{M\times N}(x, y) \asymp d_M(x_M, y_M) + d_N(x_N, y_N) \,,$$

   by Proposition 3.1 applied to the metric $g_{M\times N}$ and $g_M \times g_N$.

# 4 Submanifolds

**Lemma 4.1.** *Let $M \subset \mathbb{R}^n$ be a smooth submanifold, equipped with a Riemannian metric $g$. Consider the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$. Let $K \subset M$ be compact. Then*

$$d_g(x, y) \asymp_K \|x - y\|$$

*for $x, y \in K$.*

*Proof.* One estimate is immediate: Consider the restriction $g_{\mathrm{Eucl}}|_M$ of the Euclidean metric. We have

$$d_{g_{\mathrm{Eucl}}|_M}(x,y) \geq d_{g_{\mathrm{Eucl}}}(x,y) = \|x-y\|$$

and Proposition 3.1 implies $d_g(x,y) \asymp d_{g_{\mathrm{Eucl}}|_M}(x,y)$. Hence

$$d_g(x,y) \gg \|x-y\|$$

We now prove the reverse estimate. Let $m = \dim M$. Every $x \in M$ has an open neighborhood $U$ for which there exists an open set $V \subset \mathbb{R}^n$ and a diffeomorphism $\phi : V \to W \subset \mathbb{R}^n$ such that $V \cap M = U$ and $\phi(U) = W \cap (\mathbb{R}^m \times \{0\})$. We may assume that $\phi(U)$ is convex. We may choose an open neighborhood $U' \subset U$ of $x$, relatively compact in $U$ and such that for $y, z \in U'$ there exists a geodesic of length $d_g(y,z)$ contained in $U$. We may assume that the convex closure of $U'$ in $\mathbb{R}^n$ is contained in $V$ and relatively compact in $V$. Among those $U'$ we may extract a finite cover of $K$. By the Lebesgue covering lemma, there exists $\delta > 0$ such that when $x, y \in K$ with $d_g(x,y) \leq \delta$, there exists an open set $U'$ of the cover such that $x, y \in U'$. We restrict our attention to $x, y \in K$ with $d_g(x,y) \leq \delta$. For the other points, we may take the implicit constant

$$\max_{\substack{x,y\in K\\ d_g(x,y)\geq\delta}} \frac{d_g(x,y)}{\|x-y\|}.$$

Now take $\phi$ and $U' \subset U$ as above with $x, y \in U'$. Consider the pushforward $\phi_*(g|_U)$ and the Euclidean metric $g_{\mathrm{Eucl}}|_{\phi(U)}$ on $\phi(U)$. By Proposition 3.1 applied to the relatively compact set $\phi(U') \subset \phi(U)$,

$$\begin{aligned} d_g(x,y) &= d_{g|_U}(x,y) \\ &= d_{\phi_*(g|_U)}(\phi(x),\phi(y)) \\ &\asymp d_{\mathrm{Eucl}\,|_{\phi(U)}}(\phi(x),\phi(y)) \\ &= \|\phi(x)-\phi(y)\| \end{aligned}$$

because $\phi(U)$ is convex. By the mean value theorem, there exists $z \in [x,y]$ such that $\|\phi(x)-\phi(y)\| \leq |(\nabla\phi)(z)\cdot(x-y)|$. Let $L$ be the convex closure of $U'$ in $\mathbb{R}^n$. Then $z \in L$. By assumption, $L$ is relatively compact in $V$, so that

$$\begin{aligned} d_g(x,y) &\asymp \|\phi(x)-\phi(y)\| \\ &\leq \left(\sup_{z\in L}\|(\nabla\phi)(z)\|\right)\cdot\|x-y\|. \end{aligned}$$

Because there are only finitely many choices of $(\phi, U, U')$ to be considered, this settles the estimate when $d_g(x,y) \leq \delta$. ☐

**Proposition 4.2.** *Let $M$ be a Riemannian manifold and $N$ a submanifold, equipped with a Riemannian metric. Let $K \subset N$ be compact. Then*

$$d_N(x,y) \asymp d_M(x,y)$$

*for $x, y \in K$.*

*Proof.* We will reduce to Lemma 4.1 using charts of $M$. Call $g_M$ and $g_N$ the metrics of $M$ and $N$. Then one estimate follows from Proposition 3.1:

$$\begin{aligned} d_M(x,y) &\leq d_{g_M|_N}(x,y) \\ &\asymp d_N(x,y)\,. \end{aligned}$$

We now prove the other estimate. As in the proof of Proposition 3.1, the case of points at $M$-distance bounded away from 0 is immediate. We may thus suppose $x$ and $y$ lie in one of finitely many compact sets $L \subset N$ contained in an open subset $U \subset M$ that is diffeomorphic to an open subset of $\mathbb{R}^m$. We may also assume that $L$ is small enough so that the $M$-distance and $N$-distance between points of $L$ are realized within $U$. Call the diffeomorphism $\phi$. By Lemma 4.1 applied to the submanifold $\phi(U \cap N) \subset \mathbb{R}^m$ with the pushforward metric $(\phi|_{U\cap N})_*(g_N|_U)$ and the compact subset $\phi(L) \subset \phi(U \cap N)$,

$$\begin{aligned} d_N(x,y) &= d_{U\cap N}(x,y) \\ &= d_{\phi(U\cap N)}(\phi(x),\phi(y)) \\ &\asymp \|\phi(x)-\phi(y)\|\,. \end{aligned}$$

By Lemma 4.1 applied to the (codimension 0) submanifold $\phi(U) \subset \mathbb{R}^n$, the metric $g_{\mathrm{Eucl}}|_{\phi(U)}$ and the compact subset $\phi(L)$,

$$\|\phi(x)-\phi(y)\| \asymp d_{\mathrm{Eucl}\,|_{\phi(U)}}(\phi(x),\phi(y))\,.$$

By Proposition 3.1 applied to the manifold $\phi(U)$, the Euclidean metric and the pushforward $\phi_* g_M$, and the compact set $\phi(L)$,

$$\begin{aligned} d_M(x,y) &= d_U(x,y) \\ &= d_{\phi(U)}(\phi(x),\phi(y)) \\ &\asymp d_{\mathrm{Eucl}\,|_{\phi(U)}}(\phi(x),\phi(y))\,, \end{aligned}$$

which completes the proof. □

# 5 Covering maps

**Proposition 5.1.** *Let $f : M \to N$ be a smooth map between Riemannian manifolds. Let $K \subset M$ be compact. Then*

$$d_N(f(x),f(y)) \ll_K d_M(x,y)$$

*for $x, y \in K$.*

*Proof.* Similar to the proof of Proposition 3.1, using Proposition 1.2 together with the fact that the differential $Df$ increases norms of tangent vectors locally by at most a constant. □

**Proposition 5.2.** *Let $p : M \to N$ be a local diffeomorphism between smooth Riemannian manifolds. Let $K \subset M$ be compact and $x, y \in p(K)$. Let $L \supset K$ be a compact neighborhood of $K$. Let $\sigma : p(K) \to K$ be an arbitrary section of $p$. Then*

$$d_N(x,y) \asymp_{K,L} \min_{i,j} d_M(p|_L^{-1}(x), p|_L^{-1}(y)) \asymp_{K,L} \min_j d_M(\sigma(x), p|_L^{-1}(y))$$

*for $x, y \in p(K)$.*

*Proof.* Denote $p|_L^{-1}(x) = \{x_0, \ldots, x_n\}$ and $p|_L^{-1}(y) = \{y_0, \ldots, y_m\}$, with $\sigma(x) = x_0 \in K$. The estimate

$$d_N(x,y) \ll \min_{i,j} d_M(x_i, y_j)$$

follows from Proposition 5.1, for which we don't need to assume that $L$ is a neighborhood of $K$. We trivially have

$$\min_{i,j} d_M(x_i, y_j) \leq \min_j d_M(x_0, y_j)\,.$$

It remains to show that

$$\min_j d_M(x_0, y_j) \ll_{K,L} d_N(x,y)\,.$$

Cover $L$ by finitely many open sets $(U_i)_{i \in I}$ on which $p$ is injective. When $d_N(x,y) \geq r > 0$ we may take as implicit constant the maximum of the upper semi-continuous function

$$(x,y) \mapsto r^{-1} \cdot \max_{\substack{i \in I \\ x \in p(U_i)}} \min_{\substack{j \\ y \in p(U_j)}} d_M\left(p|_{U_i}^{-1}(x), p|_{U_j}^{-1}(y)\right)$$

on the compact set $\{(x,y) \in p(K) \times p(K) : d_N(x,y) \geq r\}$. Here again we do not use that $L$ is a neighborhood of $K$.

Because $L$ is a neighborhood of $K$, we may cover cover $K$ by finitely many open sets $V_i \subset L$ that are relatively compact in an open set $W_i \subset M$ on which $p$ is injective. We may assume that each $V_i$ is small enough so that the distances between points of $p(V_i)$ are realized in $p(W_i)$. There exists $r > 0$ such that for every $a \in K$ there exists $V_i \ni a$ with $B_N(p(a), r) \subset p(V_i)$. Indeed, if some $p(V_i)$ equals $N$, this is trivial. Otherwise, we may take $r$ to be the minimum of the continuous positive function

$$a \mapsto \max_i d_N(p(a), N - p(V_i))$$

on the compact set $K$. As remarked earlier, we may now assume that

$$d_N(x,y) < r\,.$$

Let $V_i \ni x_0$ be such that $B_N(x,r) \subset p(V_i)$. Then $y \in p(V_i)$. By Proposition 5.1 applied to the smooth map $p|_{W_i}^{-1}$ and the relatively compact subset $p(V_i) \subset p(W_i)$, we have

$$\begin{aligned} d_N(x,y) &= d_{p(W_i)}(x,y) \\ &\gg d_M(x_0, p|_{W_i}^{-1}(y)) \\ &\geq \min_j d_M(x_0, y_j) \end{aligned}$$

because $p|_{W_i}^{-1}(y) \in V_i \subset L$. □

**Proposition 5.3.** *Let $p : M \to N$ be a finite degree smooth covering map between smooth Riemannian manifolds. Let $K \subset N$ be compact. Then*

$$d_N(x,y) \asymp_K d_M(p^{-1}(x), p^{-1}(y))$$

*for $x, y \in K$.*

*Proof.* By applying Proposition 5.2 to the compact set $K' = p^{-1}(K)$ and any compact neighborhood $L \supset K'$. □

**Example 5.4.**

1. With the notations of Proposition 5.2, let $M = \mathbb{R}$, $N = \mathbb{R}/\mathbb{Z}$ and $K = L = [0,1] \subset M$. The projection $p : M \to N$ is a covering map but it is not true that

   $$d_N(x,y) \asymp \min_{i,j} d_M(x_i, y_j)$$

   when $x, y \in p(K)$. For example when $x = p(0)$ and $y \to p(1)^-$. So we cannot omit the condition that $L$ is a neighborhood of $K$.

2. Let $M = \mathbb{R}$, $N = \mathbb{R}/\mathbb{Z}$, $K = [0, 0.99]$ and $L = [-0.5, 1]$. Let $x = p(0)$, $x_0 = 1 \in L - K$ and $y \to p(0)^+$. Then it is not true that

   $$d_N(x,y) \asymp \min_j d_M(x_0, y_j)\,.$$

   So we cannot omit the condition that $\sigma(x) \in K$.

3. With the same notations, let $M = N = \mathbb{C}$ and $p(z) = z^2$. Then $p$ is open; it is a ramified covering map. But it is not true that

   $$d_N(0, p(z)) \asymp \min(d_M(0,z), d_M(0,-z))$$

   as $z \to 0$: the LHS is $\asymp |z|^2$, the RHS is $\asymp |z|$. We cannot omit the condition that $p$ is everywhere a local diffeomorphism.

4. Equip $\mathrm{GL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$ with Riemannian metrics and $M_2(\mathbb{R})$ with a matrix norm. Then for $K \subset \mathrm{SL}_2(\mathbb{R})$ compact and $x, y \in K$ we have

   $$\begin{aligned} d_{\mathrm{PSL}_2}(x,y) &\asymp \min(d_{\mathrm{SL}_2}(x,y), d_{\mathrm{SL}_2}(x,-y)) \\ &\asymp \min(d_{\mathrm{GL}_2}(x,y), d_{\mathrm{GL}_2}(x,-y)) \\ &\asymp \min(\|x-y\|, \|x+y\|)\,, \end{aligned}$$

   by respectively Proposition 5.3 and two times Proposition 4.2.