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A brief and relatively terse account of continued fractions. Periodicity of continued fractions of Laurent series is not discussed.

Contents

1 Real continued fractions

Definition 1.1. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Its sequence of* ***complete quotients*** $(\alpha_n)_{n \geqslant 0}$ *is defined recursively by:*

$$\alpha_0 = x, \quad \alpha_{n+1} = \frac{1}{\{\alpha_n\}} \qquad (n \geqslant 0),$$

The ***continued fraction*** *of x is the sequence of* ***partial quotients*** $(a_n)_{n \geqslant 0}$ *defined by:*

$$a_n = \lfloor \alpha_n \rfloor$$

Thus

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n} \quad \text{and} \quad \alpha_n = a_n + \frac{1}{\alpha_{n+1}} \qquad (n \geqslant 0)$$

The second identity implies

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_n + \cfrac{1}{\alpha_{n+1}}}}} = [a_0, a_1, \ldots, a_n, \alpha_{n+1}]$$

and the freshly introduced $[\cdot]$-notation satisfies/is recursively defined by:

$$[a,b] = a + \frac{1}{b}, \quad [a_0,\dots,a_k,a_{k+1},\dots,a_n] = [a_0,\dots,a_k,[a_{k+1},\dots,a_n]]$$

but is not associative. Note that $\alpha_n > a_n \geqslant 1$ for $n > 0$.

Definition 1.2. *The sequences of* ***numerators*** *and* ***denominators*** *are defined informally*[1] *by writing*

$$p_n/q_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \frac{1}{a_n}}} = [a_0, a_1, \dots, a_n]$$

as a simple fraction, formally below. $p_n/q_n = [a_0,\dots,a_n]$ *is called the* n*th* ***convergent****.*

If we define the usual group action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

so that

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \cdot z = a + \frac{1}{z} = [a,z]$$

then

$$\alpha_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot \alpha_{n+1} \qquad (n \geqslant 0)$$

so

$$x = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot \alpha_{n+1}$$

We want p_n/q_n to be the result of this when $\alpha_{n+1} = \infty$, that is, we define (p_n, q_n) by

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & * \\ q_n & * \end{pmatrix} \qquad (n \geqslant 0)$$

and from

$$\begin{pmatrix} p_n & * \\ q_n & * \end{pmatrix} = \begin{pmatrix} p_{n-1} & * \\ q_{n-1} & * \end{pmatrix} \cdot \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

we have

$$\begin{pmatrix} p_n & * \\ q_n & * \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \qquad (n \geqslant 1)$$

(and also for $n = 0$ if we define $(p_{-1}, q_{-1}) = (1, 0)$) and hence the recurrences

$$\begin{cases} p_n = p_{n-1}a_n + p_{n-2} \\ q_n = q_{n-1}a_n + q_{n-2} \end{cases} \qquad (n \geqslant 1)$$

[1] We could but don't want to formalize this using the field of rational functions $\mathbb{Q}(a_k : k \leqslant n)$ and the fact that $\mathbb{Z}[a_k : k \leqslant n]$ is a UFD.

and also for $n = 0$ if we define $(p_{-2}, q_{-2}) = (0, 1)$. [2] Explicitly:

n	-2	-1	0	1	2	3
p_n	0	1	a_0	$a_0a_1 + 1$	$a_0a_1a_2 + a_2 + a_0$	$\ldots$
q_n	1	0	1	a_1	$a_1a_2 + 1$	$\ldots$

Proposition 1.3. $\gcd(p_n, q_n) = 1$

Proof. Follows from

$$\det \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \det \left(\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \right) = (-1)^{n+1}$$

□

1.1 Convergence

By the recurrence $q_n = q_{n-1}a_n + q_{n-2}$, all $q_n \geqslant 0$ because the only a_n that can be negative is a_0, which is canceled by $q_{-1} = 0$. Because $a_n \geqslant 1$ for $n \geqslant 1$, $q_n \geqslant q_{n-1} + q_{n-2}$ for $n \geqslant 1$ (and indeed for $n = 0$ as well). In summary:

Proposition 1.4. *The denominators* (q_n) *satisfy:*

1. $q_n \geqslant 0$ *for* $n \geqslant -2$
2. (q_n) *is increasing for* $n \geqslant -1$
3. $q_n \geqslant 1$ *for* $n \geqslant 0$
4. (q_n) *is strictly increasing for* $n \geqslant 1$

Proof. 1. See above. 2. From $q_n \geqslant q_{n-1} + q_{n-2}$. 3. From 2 and $q_0 = 1$. 4. From $q_n \geqslant q_{n-1} + q_{n-2}$ and $q_0, q_1, \ldots \geqslant 1$. □

Note that q_n grows exponentially, since by induction $q_n \geqslant F_{n+1}$, the $(n+1)$th Fibonacci number (starting $(F_{-1}, F_0, F_1, F_2) = (1, 0, 1, 1)$). Equality occurs for all n if and only if $a_n = 1$ for all $n \geqslant 1$. (Which may occur; see next paragraph.)

Proposition 1.5.

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

[2] This extended definition of (p_n, q_n) makes that

$$\prod_{k=0}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

also holds for $n = -1$, if we agree that an empty product is the identity matrix. This is not a coincidence, since the extended definition of (p_n, q_n) is based on their recurrence relation, which in turn is based on the above product identity; also for $n = -1$.

Proof. For $n \geqslant 1$,

$$\det \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \det \left(\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \right) = (-1)^{n+1}$$

hence $p_n/q_n < p_{n-1}/q_{n-1}$ for $n \geqslant 0$ even (because $q_n > 0$). It remains to show that the even convergents are increasing and the odd ones decreasing. We have

$$\begin{cases} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \cdot \begin{pmatrix} a_n \\ 1 \end{pmatrix} \\ \begin{pmatrix} p_{n-2} \\ q_{n-2} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases} \qquad (n \geqslant 0)$$

so

$$\begin{pmatrix} p_n & p_{n-2} \\ q_n & q_{n-2} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \cdot \begin{pmatrix} a_n & 0 \\ 1 & 1 \end{pmatrix} \qquad (n \geqslant 0)$$

hence

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$$

and the conclusion follows since $a_n > 0$ for $n > 0$. □

Proposition 1.6.

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < x < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

Proof. We have, for $n \geqslant 0$,

$$x = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot \alpha_{n+1} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \cdot \alpha_{n+1}$$

and $\alpha_{n+1} > 0$. For $a \in \mathbb{R}$, the function

$$z \mapsto \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \cdot z = a + \frac{1}{z}$$

is strictly decreasing for $z > 0$. Thus for n even,

$$z \mapsto \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \cdot z$$

is strictly decreasing for $z > 0$ and has limit p_n/q_n for $z \longrightarrow \infty$. For n odd, the other way around. □

Theorem 1.7. $p_n/q_n \longrightarrow x$

Proof. It suffices to show that $p_{n+1}/q_{n+1} - p_n/q_n \longrightarrow 0$. This follows from

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \left| \frac{(-1)^n}{q_n q_{n+1}} \right| \leqslant \frac{1}{q_n^2} \qquad (n \geqslant 0)$$

and the fact that $q_n \longrightarrow \infty$. □

We may thus write/define

$$[a_0, a_1, a_2, \dots] = x$$

Corollary 1.8. *For all $n \geqslant 0$,*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$$

Proof. Follows from the previous proof. Equality cannot occur because x is irrational. □

1.2 Good approximations

Proposition 1.9. *For all $n \geqslant 0$ there is $m \in \{n, n+1\}$ with*

$$\left| x - \frac{p_m}{q_m} \right| < \frac{1}{2q_m^2}$$

Proof. For n even,

$$\frac{p_n}{q_n} < x < \frac{p_{n+1}}{q_{n+1}}$$

and

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}} \leqslant \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

For n odd, same story. □

The inequality

$$\frac{1}{q_n q_{n+1}} \leqslant \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

is tight only for $q_n \approx q_{n+1}$. But as noted above, we expect q_{n+1} to be at least about φq_n, where $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ is the golden ratio.

Proposition 1.10 (Hurwitz). *For all $n \geqslant 0$ there is $m \in \{n, n+1, n+2\}$ with*

$$\left| x - \frac{p_m}{q_m} \right| < \frac{1}{\sqrt{5} q_m^2}$$

Proof. We want to exploit the inequality $q_{n+2} \geqslant q_{n+1} + q_n$ to obtain an estimate of the form $q_{n+1} \geqslant C q_n$ with a good constant C. We're interested in q_{n+1}/q_n, so we rewrite the recurrence as

$$\left(\frac{q_{n+2}}{q_{n+1}} - 1 \right) \frac{q_{n+1}}{q_n} \geqslant 1$$

Since $(\varphi - 1)\varphi = 1$, either $\frac{q_{n+2}}{q_{n+1}} \geqslant \varphi$ or $\frac{q_{n+1}}{q_n} \geqslant \varphi$. Let $k \in \{n, n+1\}$ with $q_{k+1} > \varphi q_k$. (Strictly, because $\varphi \notin \mathbb{Q}$.) We want a good constant $D > 0$ with

$$\frac{1}{q_k q_{k+1}} < \frac{1}{D q_k^2} + \frac{1}{D q_{k+1}^2}$$

By completing the square this is equivalent to

$$\left(\frac{q_{k+1}}{q_k} - \frac{D}{2}\right)^2 > \frac{D^2}{4} - 1$$

which is satisfied for $q_{k+1}/q_k > \varphi$ and $D = \sqrt{5}$. □

Similarly, for each $n \geqslant 0$ there is $k \in \{n, n+1\}$ with $q_{k+1} > \frac{a_n + \sqrt{a_n^2+4}}{2} q_k$ and $m \in \{k, k+1\}$ with

$$\left|x - \frac{p_m}{q_m}\right| < \frac{1}{\sqrt{a_n^2 + 4} \cdot q_m^2}$$

Large partial quotients give good approximations, so in some sense φ is the most irrational number:

Proposition 1.11. *1. φ has continued fraction* $(1, 1, \ldots)$

2. For all $\varepsilon > 0$ there exist at most finitely many rationals p/q with

$$\left|\varphi - \frac{p}{q}\right| < \frac{1}{(\sqrt{5} + \varepsilon)q^2}$$

Proof. 1. In the next paragraph we show that $[1, 1, \ldots]$ converges to a positive real number and that $[1, \ldots] = 1 + 1/[1, \ldots]$. Consequently, $\varphi = [1, \ldots]$. We also show that this implies that the continued fraction of φ, as constructed in the first paragraph, is $(1, 1, \ldots)$. Alternatively, one may observe that $\{\varphi\} = 1/\varphi$ implies $\alpha_n = \varphi$ for all n, so $a_n = 1$ for all n.

2. $|\varphi - \frac{p}{q}| < 1/2q^2$ implies p/q is a convergent. The convergents to φ are precisely F_{n+1}/F_n (from the recurrence relations), and using Binet's formula $F_n = (\varphi^n - (-\varphi)^{-n})/\sqrt{5}$ we find

$$|1 - (-1)^n \varphi^{-2n}| < \frac{\sqrt{5}}{\sqrt{5} + \varepsilon}$$

In particular, $n \ll \log \varepsilon$ (and is even). □

1.3 The inverse procedure

We define arbitrary continued fractions and study their convergence.

Definition 1.12. *A* ***simple continued fraction*** *is a sequence of integers $(a_n)_{n \geqslant 0}$ with $a_n \geqslant 1$ for $n \geqslant 1$, called sequence of* ***partial quotients****. The nth* ***convergent*** *is $[a_0, \ldots, a_n]$, and the sequences of* ***numerators*** *and* ***denominators*** *are defined by*

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & * \\ q_n & * \end{pmatrix} \qquad (n \geqslant 0)$$

so that $p_n/q_n = [a_0, \ldots, a_n]$.

As long as we don't talk about convergence, they have the same properties: the recurrence relation for (p_n, q_n), Proposition 1.4, Proposition 1.5 and the inequalities

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \left|\frac{1}{q_n q_{n+1}}\right| \leqslant \frac{1}{q_n^2} \qquad (n \geqslant 0)$$

but not Proposition 1.6, since it mentions the limit.

Theorem 1.13. *The sequence of convergents converges.*

Proof. By

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| \leqslant \frac{1}{q_n^2} \qquad (n \geqslant 0)$$

and the fact that $q_n \geqslant F_{n+1}$ grows exponentially, it is Cauchy.[3] □

As before, we may define

$$[a_0, a_1, a_2, \ldots]$$

to be the limit. We have that

$$[a_0, a_1, a_2, \ldots] \text{ converges} \Leftrightarrow [a_{k+1}, a_{k+2}, \ldots] \text{ converges}$$

in which case

$$[a_0, a_1, a_2, \ldots] = [a_0, \ldots, a_k, [a_{k+1}, a_{k+2}, \ldots]]$$

Proposition 1.14.

$$a_0 < [a_0, \ldots] < a_0 + 1$$

Proof. (When the continued fraction is constructed from a real number, this follows from the definition of $(a_n)_{n\geqslant 0}$. The interest is that it holds for real numbers constructed from a continued fraction.) The first inequality follows by taking limits in

$$a_0 = \frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots$$

The second from

$$[a_0, \ldots] = a_0 + \frac{1}{[a_1, \ldots]} < a_0 + 1$$

by applying the first to $[a_1, \ldots]$. □

At last we get rid of an annoying ambiguity:

Theorem 1.15. $[a_0, \ldots]$ *is irrational, and its continued fraction is* $(a_n)_{n\geqslant 0}$.

Proof. Let $(b_n)_{n\geqslant 0}$ be its continued fraction.[4] From

$$[a_0, \ldots] = [b_0, \ldots]$$

and Proposition 1.14 we have $a_0 = b_0$. From

$$[a_0, [a_1, \ldots]] = [b_0, [b_1, \ldots]]$$

and the injectivity of $z \mapsto a + 1/z$ we have $[a_1, \ldots] = [b_1, \ldots]$, and we proceed by induction. □

[3] As in the proof of Banach's/Picard's fixed point theorem for contraction mappings.

[4] That it is irrational, in fact follows from the proof, but we don't want to hide the quick argument behind too many details.

1.4 A homomorphism

We study how continued fractions behave with respect to the topology of $\mathbb{R} \setminus \mathbb{Q}$.

Proposition 1.16. *The map which sends an irrational number to its continued fraction in* $\mathbb{Z} \times \mathbb{N}^{\mathbb{N}}$ *(with the product topology) is continuous.*

Proof. A sub-basis[5] of $\mathbb{Z} \times \mathbb{N}^{\mathbb{N}}$ consists of the *cylinders*, inverse images of an $a \in \mathbb{Z}$ or $a \in \mathbb{N}$ by one of the projections. Such a cylinder corresponds to

$$\bigcup_A A \cdot (a, a+1) \setminus \mathbb{Q}$$

which is open in $\mathbb{R} \setminus \mathbb{Q}$, where the union is taken over a certain set of matrices in $\mathrm{GL}_2(\mathbb{Z})$.[6] □

Proposition 1.17. *It is open.*

Proof. An interval $(a, a+1) \subset \mathbb{R} \setminus \mathbb{Q}$ with $a \in \mathbb{Z}$ is sent to the cylinder $(a, *, *, \ldots)$. If $a > 0$, the set

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot (a, a+1)$$

is sent to the finite intersection of cylinders given by $(a_0, \ldots, a_n, a, *, *, \ldots)$. Let $U \subset \mathbb{R} \setminus \mathbb{Q}$ be open. The hope is/we have to show that for every $x \in U$ with continued fraction $(a_0, \ldots)$ there is $n \in \mathbb{N}$ such that U contains

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot (a_{n+1}, a_{n+1} + 1)$$

Because $z \mapsto a + \frac{1}{z}$ is decreasing for $z > 0$, the above is an interval of length

$$\left| \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot a_{n+1} - \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot (a_{n+1} + 1) \right|$$

We'd like this to go to 0 (or at least have lim inf equal to 0). Keeping in mind that the action of $A \in \mathrm{GL}_2(\mathbb{Z})$ on $z \in \mathbb{R}$ is also given by x/y if

$$A \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

[5] That is, open sets are unions of finite intersections of the sub-basis elements.

[6] To be precise (but it really doesn't matter), the matrices of the form

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

with $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{N}$ for $k > 0$ and $n + 1$ being the index determining the cylinder.

the length equals

$$\begin{aligned}\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n(a_{n+1}+1)+p_{n-1}}{q_n(a_{n+1}+1)+q_{n-1}}\right| &= \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+1}+p_n}{q_{n+1}+q_n}\right| \\ &= \left|\frac{p_{n+1}q_n - q_{n+1}p_n}{q_{n+1}(q_{n+1}+q_n)}\right| \\ &= \frac{1}{q_{n+1}(q_{n+1}+q_n)} \\ &\to 0\end{aligned}$$

□

In summary:

Theorem 1.18. *The map which sends an irrational number to its continued fraction is a homeomorphism between $\mathbb{R}\setminus\mathbb{Q}$ and $\mathbb{Z}\times\mathbb{N}^{\mathbb{N}}$.*

We also see that:

Proposition 1.19. *For all $n \geqslant 0$, the set of irrationals that have the same continued fraction as x up to the nth partial quotient form a small interval around x. If $a_{n+1} \neq 1$, this interval contains the interval with endpoints x and*

$$x + \frac{(-1)^n}{2(q_n+q_{n-1})^2}$$

1.5 Periodicity

Proposition 1.20. *Let $x \in \mathbb{R}\setminus\mathbb{Q}$. The following are equivalent:*

1. *Its sequence of partial quotients (a_n) is periodic for $n \geqslant N$.*
2. *Its sequence of complete quotients (α_n) is periodic for $n \geqslant N$.*
3. *α_N appears again in the sequence of complete quotients.*

Proof. $1 \implies 2$: because $\alpha_n = [a_n, \ldots]$. $2 \implies 1$: because $a_n = \lfloor \alpha_n \rfloor$. $2 \iff 3$: because the next complete quotient is determined from the previous one. □

We say $x \in \mathbb{R}\setminus\mathbb{Q}$ has eventually periodic continued fraction.

Theorem 1.21. *Let $x \in \mathbb{R}\setminus\mathbb{Q}$ with eventually periodic continued fraction. Then x is a quadratic algebraic number.*

Proof. Say $x = [a_0, \ldots]$ and $a_{n+k} = a_n$ for all $n > N$, then

$$y = \left(\begin{pmatrix} a_0 & 1 \\ 1 & 0\end{pmatrix} \cdots \begin{pmatrix} a_N & 1 \\ 1 & 0\end{pmatrix}\right)^{-1} \cdot x = [a_{N+1} \ldots]$$

satisfies $y = [a_{N+1}, \ldots, a_{N+k}, y]$, that is,

$$y = \begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1}\end{pmatrix}^{-1} \cdot \begin{pmatrix} p_{N+k} & p_{N+k-1} \\ q_{N+k} & q_{N+k-1}\end{pmatrix} \cdot y$$

We want the right bottom entry in this matrix to be nonzero, because then y satisfies a quadratic equation with non-vanishing second degree coefficient. Because

$$\begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1} \end{pmatrix}^{-1} \cdot \begin{pmatrix} p_{N+k} & p_{N+k-1} \\ q_{N+k} & q_{N+k-1} \end{pmatrix} = \begin{pmatrix} a_{N+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{N+k} & 1 \\ 1 & 0 \end{pmatrix}$$

and all a_{N+j} are positive, the right bottom entry is nonzero. □

We have a (right) group action of $\mathrm{GL}_2(\mathbb{Z})$ on integral binary quadratic forms, by

$$(f|A)(\mathbf{x}) = f(A\mathbf{x})$$

If M is the matrix of f, so that $f(\mathbf{x}) = \mathbf{x}^t M\mathbf{x}$, then $A^t MA$ is the matrix of $(f|A)$. Since the discriminant $\Delta(f) = -4\det M$, we have $\Delta(f|A) = \Delta(f) \cdot \det(A)^2 = \Delta(f)$. Let $\overline{f} = f(x, 1)$ denote dehomogenization. Then (x, y) with $y \neq 0$ is a root of f if and only if x/y is a root of $\overline{f}$.
The link with the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R} \setminus \mathbb{Q}$ is as follows: x is a root of $\overline{f}$ if and only if $A^{-1}x$ is a root of $\overline{(f|A)}$.
Finally, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the coefficient of x^2 in $f|A$ is

$$f\left(A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = f(a, c)$$

and the coefficient of y^2 is $f(b, d)$.

Theorem 1.22. *Let $\alpha \in \mathbb{R}$ be a quadratic irrational. Then its continued fraction is eventually periodic.*

Proof. Let α be a root of an integral quadratic polynomial $\overline{f}$, and f its homogenization. Since

$$\alpha_n = \underbrace{\begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix}^{-1}}_{M_n} x$$

$(\alpha_n, 1)$ is a root of $f|M_n$, which has discriminant $\Delta = \Delta(f)$ and has the form

$$f(p_{n-1}, q_{n-1})x^2 + [\dots]xy + f(p_{n-2}, q_{n-2})y^2.$$

If we show that these coefficients are bounded, then the complete quotients take only finitely many values, as desired. The term in xy is determined up to sign by Δ and the other coefficients, so it suffices to bound $f(p_n, q_n)$. We have $p_n/q_n = \alpha + O(1/q_n^2)$, so indeed

$$f(p_n, q_n) = q_n^2 f(p_n/q_n, 1) = O(1)$$

using Taylor approximation around α. □

2 Laurent Series

Definition 2.1. *Let k be a field. We denote $k((t^{-1}))$ the field of (finite-tailed) Laurent-series in t^{-1}.*[7] *That is, series of the form*

$$\sum_{n=-\infty}^{N} c_n t^n$$

The integer part of a series is its polynomial part, denoted $\lfloor f \rfloor$; its fractional part $\{f\}$ is what remains, so that

$$f = \lfloor f \rfloor + \{f\}$$

The field $k((t^{-1}))$ contains the field of rational functions $k(t)$, and becomes a complete ultrametric space with the absolute value

$$|f| := \begin{cases} e^{\deg f} & f \neq 0 \\ 0 & f = 0 \end{cases}$$

where $\deg f$ is the largest index with a nonzero coefficient ($f \neq 0$).[8]

Definition 2.2. *Let $f \in k((t^{-1})) \setminus k(t)$. Its sequence of **complete quotients** $(\alpha_n)_{n \geqslant 0}$ is defined recursively by:*

$$\alpha_0 = f, \quad \alpha_{n+1} = \frac{1}{\{\alpha_n\}} \qquad (n \geqslant 0),$$

*The **continued fraction** of f is the sequence of **partial quotients** $(a_n)_{n \geqslant 0}$ defined by:*

$$a_n = \lfloor \alpha_n \rfloor$$

Thus

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n} \quad \text{and} \quad \alpha_n = a_n + \frac{1}{\alpha_{n+1}} \qquad (n \geqslant 0)$$

The second identity implies

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \, \cfrac{1}{a_n + \cfrac{1}{\alpha_{n+1}}}}} = [a_0, a_1, \ldots, a_n, \alpha_{n+1}]$$

Note that $\deg \alpha_n = \deg a_n \geqslant 1$ for $n > 0$.

Definition 2.3. *The sequences of **numerators** and **denominators** are defined by*

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & * \\ q_n & * \end{pmatrix} \qquad (n \geqslant 0)$$

*and $p_n/q_n = [a_0, \ldots, a_n]$ is called the nth **canonical convergent**.*

[7] While there is no difference with considering $k((t))$ instead, the analogue of *integral part* is more intuitive this way.

[8] e is an arbitrary choice. We could as well formulate everything in terms of the valuation at t, without defining an absolute value.

They satisfy the same algebraic properties as those for real continued fractions, and we also define

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as before.

2.1 Convergence

Proposition 2.4. *Let $f \in k((t^{-1})) \setminus k(t)$.*

1. $\deg q_{n+1} = \deg q_n + \deg a_{n+1}$ *for* $n \geqslant 0$
2. $\deg q_{n+1} > \deg q_n$ *for* $n \geqslant 0$
3. p_n/q_n *converges*
4. *to* f

Proof. 1. We have $q_{n+2} = q_{n+1}a_{n+2} + q_n$. From $\deg a_n > 0$ if $n > 0$ and $(q_1, q_0, q_{-1}) = (a_1, 1, 0)$ it follows by induction that $\deg q_{n+1} = \deg q_n + \deg a_{n+1}$ for $n \geqslant 0$. 2. From 1. 3. From $p_n/q_n - p_{n+1}/q_{n+1} = \pm 1/q_n q_{n+1}$ the sequence of convergents is Cauchy because the metric is ultrametric. 4. We have

$$f = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \cdot \alpha_{n+1} \implies \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix} \cdot f = \pm \alpha_{n+1}$$

which, together with $q_0 f - p_0 = \{f\} = 1/\alpha_1$ implies

$$q_n f - p_n = \frac{\pm 1}{\alpha_1 \cdots \alpha_{n+1}} \qquad (n \geqslant 0)$$

and thus $f - p_n/q_n \to 0$. Note that this also works in the real case, but only gives the bound $|f - p_n/q_n| < 1/q_n$ instead of $1/q_n^2$. □

Proposition 2.5. *Let $(a_0, \ldots)$ be any continued fraction of polynomials, with* $\deg a_n > 0$ *for $n > 0$. It converges to an irrational Laurent series, whose continued fraction is again $(a_0, \ldots)$.*

Proof. The condition $\deg a_n > 0$ ensures that $\deg q_{n+1} > \deg q_n$ for $n \geqslant 0$, as before. It converges because $p_n/q_n - p_{n+1}/q_{n+1} = \pm 1/q_n q_{n+1}$. As for real numbers, the key is to show

$$\lfloor [a_0, \ldots] \rfloor = a_0 = p_0/q_0$$

By ultrametricity, $|p_0/q_0 - p_n/q_n| \leqslant 1/|q_0 q_1| < 1$ for all n. Taking limits, $|a_0 - [a_0, \ldots]| < 1$. By induction, $[a_0, \ldots]$ is irrational and its continued fraction is $(a_0, \ldots)$. □

Similarly, now that we now that $p_n/q_n \to f$, we can take limits in $|p_n/q_n - p_m/q_m| \leqslant 1/|q_n q_{n+1}|$ $(m \geqslant n)$ to obtain

$$\left| f - \frac{p_n}{q_n} \right| \leqslant \frac{1}{|q_n q_{n+1}|} \qquad (n \geqslant 0)$$

This also follows more directly from

$$q_n f - p_n = \frac{\pm 1}{\alpha_1 \cdots \alpha_{n+1}} \qquad (n \geqslant 0)$$

by noting that $\deg q_n = \deg(a_1 \cdots a_n) = \deg(\alpha_1 \cdots \alpha_n)$.

2.2 A homomorphism

As for real numbers, we prove:

Theorem 2.6. *The map which sends an irrational Laurent series to its continued fraction is a homeomorphism between $k((t^{-1})) \setminus k(t)$ and $k[t] \times k[t]_{>0}^{\mathbb{N}}$.*

Corollary 2.7. *For any countable field k, $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic*[9] *to $k((t^{-1})) \setminus k(t)$.*

Note that for k uncountable, $k((t^{-1})) \setminus k(t)$ does no longer have a countable basis.

Proposition 2.8. *The map which sends an irrational Laurent series to its continued fraction in $k[t] \times k[t]_{>0}^{\mathbb{N}}$ (with the product topology, and $k[t]$ discrete) is continuous.*

Proof. A sub-basis of $k[t] \times k[t]_{>0}^{\mathbb{N}}$ consists of the *cylinders*, inverse images of an $f \in k[t]$ or $f \in k[t]_{>0}$ by one of the projections. Such a cylinder corresponds to

$$\bigcup_A A \cdot B(f, 1) \setminus k(t)$$

which is open in $k((t^{-1})) \setminus k(t)$, the union taken over a certain set of matrices in $\mathsf{GL}_2(k[t])$. □

Proposition 2.9. *It is open.*

Proof. A ball $B(f, 1) \subset k((t^{-1})) \setminus k(t)$ with $f \in k[t]$ is sent to the cylinder $(f, *, *, \ldots)$. If $\deg f > 0$, the set

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot B(f, 1)$$

is sent to the finite intersection of cylinders given by $(a_0, \ldots, a_n, f, *, *, \ldots)$. Let $U \subset k((t^{-1})) \setminus k(t)$ open and $f \in U$ with continued fraction $(a_0, \ldots)$. Does U contain

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot B(a_{n+1}, 1)$$

for some n? Let $n \in \mathbb{N}$ and $a_{n+1} + \delta \in B(a_{n+1}, 1)$. We want to estimate

$$\left| \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot a_{n+1} - \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdot (a_{n+1} + \delta) \right|$$

We'd like this to go to 0 uniformly in δ as $n \to \infty$. It equals

$$\begin{aligned} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n(a_{n+1} + \delta) + p_{n-1}}{q_n(a_{n+1} + \delta) + q_{n-1}} \right| &= \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+1} + \delta p_n}{q_{n+1} + \delta q_n} \right| \\ &= |\delta| \left| \frac{p_{n+1} q_n - q_{n+1} p_n}{q_{n+1}(q_{n+1} + q_n)} \right| \\ &< \frac{1}{|q_{n+1}(q_{n+1} + q_n)|} \\ &\to 0 \end{aligned}$$

□

[9] But not isometric!