Spectral theory of automorphic forms and analytic continuation of Eisenstein series

Bart MICHELS

Advisor: F. Brumley

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Université Pierre et Marie Curie – Sorbonne Université

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Bart Michels
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1 Introduction

Real analytic Eisenstein series for $\mathsf{PSL}_2(\mathbb{R})$ in their general form and their meromorphic continuation, were first studied by Selberg, with the aim of determining the decomposition of the spectrum of the Laplace operator Δ on a finite-volume hyperbolic surface: it measures the default of diagonalizability of Δ. His ultimate objective was to prove a *trace formula*, with applications to representation theory of $\mathsf{PSL}_2(\mathbb{R})$.

Selberg's work was later generalized to other groups by Langlands, and various proofs were published by Bernstein, Selberg, Colin de Verdière and others. We will limit our attention to Eisenstein series on $\mathsf{PSL}_2(\mathbb{R})$ whose associated character of the maximal compact subgroup $\mathsf{PSO}_2(\mathbb{R})$ is trivial. That is, they are functions on the hyperbolic plane $\mathbf{H} \cong \mathsf{PSL}_2(\mathbb{R})/\,\mathsf{PSO}_2(\mathbb{R})$, which are invariant under the action of a lattice $\Gamma \subseteq \mathsf{PSL}_2(\mathbb{R})$. The Eisenstein series becomes a function $E(w,s)$ on $\mathbf{H}\times\mathbb{C}$, defined for suitable s.

Throughout the text, we will assume that Γ has only one cusp located at ∞ (for definitions, see the next section). Without this assumption, the notations become heavier and the proofs slightly more technical. This allows us to focus more on the methods used to prove meromorphic continuation, and the fundamental difficulties that arise.

One question we need to ask ourselves, is what it means to analytically (or meromorphically) continue the Eisenstein series: it is a function of two variables, so one can for example interpret holomorphy to mean that $E(w,s)$ is holomorphic for fixed w, or the stronger property that it is holomorphic as a function on U with values in the Fréchet space of smooth functions $C^\infty(\mathbf{H})$. That is: that the limit

$$\lim_{s\to s_0} \frac{E(w,s)-E(w,s_0)}{s-s_0}$$

exists as an element of $C^\infty(\mathbf{H})$ (Definition B.4). One can also wonder about additional regularity conditions. We will show:

Theorem 1.1. The real analytic Eisenstein series as defined in (4.1) has a C^∞-meromorphic continuation to $\mathbb{C}$, which is jointly smooth away from poles, as a function on $\mathbf{H}\times\mathbb{C}$. It satisfies a functional equation of the form

$$E(w,1-s)=\phi(s)E(w,s)$$

for some $\mathbb{C}$-valued meromorphic function ϕ satisfying $\phi(s)\phi(1-s)=1$.

Here, C^∞-meromorphy can equivalently be formulated in terms of Laurent-expansions (Appendix B.4) or as C^∞-holomorphy up to a $\mathbb{C}$-valued meromorphic factor (5.29). We will give two proofs: one is due to Selberg and uses the theory of Fredholm integral equations. The other uses Bernstein's continuation principle, although in the second proof we only show the existence of a meromorphic continuation for a weaker topology on $C^\infty(\mathbf{H})$, the L^2_{loc} topology.

Minor refinements aside, my personal contribution to the study of Eisenstein series consists of a Fredholm theorem for noncompact integral operators (C.5), a study of holomorphic and meromorphic functions with values in function spaces (Appendix B), as well as a proof of a notorious finite type condition in an application of Bernstein's continuation principle (§5.3.4). It is possible that these results are not new, and are simply hard to find in the literature.

2 The hyperbolic plane

2.1 Isometries and geometry

Denote by $\mathbf{H} = \mathbf{H}^2 = \{z \in \mathbb{C} : \Im\mathrm{m}\, z > 0\}$ the upper half-plane. We will often write $\Im\mathrm{m}\, z = y$ and talk about "the function y" when what we mean is the imaginary part. We recall some classical facts about the geometry of $\mathbf{H}$.

Proposition 2.1. The group $G = \mathsf{PSL}_2(\mathbb{R})$ acts transitively and faithfully on $\mathbf{H}$ by homographies/Möbius-transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}$$

Moreover, it is the full group of holomorphic automorphisms.

Proof. That this defines a group action on $\mathbb{P}^1(\mathbb{C})$ is because the usual action $\mathsf{SL}_2(\mathbb{R}) \curvearrowright \mathbb{A}^2(\mathbb{C})$ is linear, so it descends to $\mathbb{P}^1(\mathbb{C})$, and it is trivial on the subgroup of diagonal matrices $\{\pm I\}$. That $\mathbf{H}$ is stable follows from

$$\Im\mathrm{m}(\gamma z) = \frac{\Im\mathrm{m}\, z}{|cz+d|^2} \quad , \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.2}$$

To see that the action is transitive, from the above formula we see that a fixed z can be sent to a point with any imaginary part, after which a horizontal translation can take care of the real part. That it is the full group of holomorphic automorphisms, follows from Schwarz's lemma in complex analysis. $\square$

Thus, the action $\mathsf{PSL}_2(\mathbb{R}) \curvearrowright \mathbb{P}^1(\mathbb{C})$ decomposes into three orbits:

$$\mathbb{P}^1(\mathbb{C}) = \mathbf{H} \sqcup \mathbb{P}^1(\mathbb{R}) \sqcup -\mathbf{H}$$

Proposition 2.3. The stabilizer of i is

$$K := \mathsf{PSO}_2(\mathbb{R}) = \left\{ \pm \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} / \{\pm I\}$$

Proof. This can of course be done by a direct computation (or, indeed, using Schwarz's lemma). We give a conceptual argument, which explains what's so special about the point i that we obtain the rotation group, and not some arbitrary conjugate of it. For $\gamma \in \mathsf{SL}_2(\mathbb{R})$, we have $\gamma i = \gamma$ if and only if there is $\lambda \in \mathbb{C}^\times$ with

$$\gamma \begin{pmatrix} i \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} i \\ 1 \end{pmatrix}$$

that is,

$$\gamma \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot i + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot 1 \right) = \lambda \begin{pmatrix} i \\ 1 \end{pmatrix}$$

We would like to substitute $1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $i = \begin{pmatrix} 0 & 1 \end{pmatrix}$ so that the LHS becomes $\gamma \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and interpret the RHS by looking at the action of $\lambda \in \mathbb{C}^\times$ on $\mathbb{C} \cong \mathbb{R}^2$ as an $\mathbb{R}$-linear map. Let's make this formal. We have natural isomorphisms of $\mathbb{R}$-vector spaces

$$\mathbb{C}^2 \cong \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \cong M_2(\mathbb{R})$$

where the last is obtained by looking at the dual $(\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2)^*$ as a space of bilinear forms. It sends $\begin{pmatrix} a & b \end{pmatrix}^t \otimes \begin{pmatrix} c & d \end{pmatrix}^t$ to the matrix $\begin{pmatrix} a & b \end{pmatrix}^t \begin{pmatrix} c & d \end{pmatrix}$. Under these isomorphisms, the action of a matrix $A \in \mathsf{SL}_2(\mathbb{R})$ on $\mathbb{C}^2$ translates as

$$L_A \leftrightarrow L_A \otimes \mathrm{id} \leftrightarrow L_A \otimes \mathrm{id} \leftrightarrow L_A \circ R_{I^t}$$

where L_A denotes left multiplication and R_B right multiplication. The action of a complex number $\lambda \in \mathbb{C}^\times$ is given by

$$L_{\lambda I} \leftrightarrow \mathrm{id} \otimes L_\lambda \leftrightarrow \mathrm{id} \otimes B \leftrightarrow L_I \circ R_{B^t}$$

where $B \in M_2(\mathbb{R})$ is the matrix obtained by considering multiplication by λ on $\mathbb{C} \cong \mathbb{R}^2$ as an $\mathbb{R}$-linear map. Thus B lies in the subgroup of $\mathsf{GL}_2(\mathbb{R})$ generated by rotations and homotheties, including those with negative ratio.

Back to our problem. We obtain, now formally, the equivalent condition that

$$\gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B$$

for some $B \in \pm\, \mathsf{SO}_2(\mathbb{R})$. Taking determinants shows that $\pm B$ has to be a rotation. Because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a rotation, we finally have that $\gamma i = i$ is equivalent to $\gamma \in \pm\, \mathsf{SO}_2(\mathbb{R})$. □

By (E.7) we obtain an isomorphism of homogeneous G-spaces

$$G/K \xrightarrow{\sim} \mathbf{H}$$

We now have (at least) three ways to give $\mathbf{H}$ a Riemannian structure:

1. G being a Lie group and K a compact subgroup, it admits a G-left-invariant and K-right-invariant metric, which is determined by the choice of a positive definite quadratic form q on T_eG: Extend q to a G-left-invariant metric by pushing it forward:

   $$h_g := (L_g)_* q$$

   and integrate over K to obtain a K-right-invariant one:

   $$\widetilde{h}_g = \int_K (R_k)^* h_{gk} dk$$

   Then $\widetilde{g}$ defines a G-invariant metric on $\mathbf{H}$. There are many q we could have started with, so we look for other ways to choose a metric on $\mathbf{H}$.

2. $\mathbf{H}$ inherits the Riemannian metric $dzd\bar{z}$ from the complex structure, which we rescale as

   $$\frac{dzd\bar{z}}{y^2}$$

   Here, $dz = dx - idy$ and $d\bar{z} = dx + idy$.

3. We consider $\mathbf{H}$ as the 2-dimensional hyperbolic space as defined in (D.7)(c), with metric

   $$\frac{dx^2 + dy^2}{y^2}$$

Conformal maps are holomorphic, hence we have a natural inclusion of the orientation preserving isometries:

$$\mathrm{Isom}^+(\mathbf{H}) \subseteq G$$

It is an equality, as can be seen from the second definition of the metric, the formula (2.2) for the transformation of imaginary parts, and the formula

$$\frac{d(\gamma z)}{dz} = \frac{1}{(cz+d)^2} \quad , \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.4}$$

That is, every smooth bijection which preserves angles automatically preserves lengths! Since the usual topology on G coincides with the compact-open topology, this is an equality of Lie groups, with $\mathrm{Isom}(\mathbf{H})$ given the Lie group structure from Myers–Steenrod (D.47).

The Riemannian measure of $\mathbf{H}$ becomes

$$d\mu(x,y) = \frac{dxdy}{y^2}$$

as computed in more generality in (D.43). We state some further results that will be useful in the sequel.

Proposition 2.5. The geodesics on $\mathbf{H}$ are given by half-circles that are orthogonal to the real line, including vertical lines (which can be seen as degenerate circles). Consequently, through every two points $z = (x_1, y_1)$ and $w = (x_2, y_2)$ there is a unique geodesic segment, the length of which is given by the hyperbolic distance

$$d(z,w) = \operatorname{arcosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2}\right) \tag{2.6}$$

In particular, the distance is a smooth function of the rational function

$$u(z,w) = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2} \tag{2.7}$$

Proposition 2.8. We have the bounds

$$d(z,w) \geqslant |\log(y_1/y_2)| \tag{2.9}$$

$$(x_1 - x_2)^2 \leqslant (\exp(d(z,w)) - 1) \cdot 2y_1 y_2 \tag{2.10}$$

Proof. 1. Note that arcosh is increasing, so we can ignore the contribution from the real parts, and we have

$$d(z,w) \geqslant \operatorname{arcosh}\left(\frac{1}{2}\left(\frac{y_1}{y_2} + \frac{y_2}{y_1}\right)\right) = |\log(y_1/y_2)|$$

Alternatively, let $\gamma = (\gamma_x, \gamma_y) : [0,T] \to \mathbf{H}$ be the smooth geodesic segment joining z and w. On an interval $[t_0, t_1]$ where γ_y is monotone, we have for the length:

$$\begin{aligned} L(\gamma|_{[t_0,t_1]}) &= \int_{t_0}^{t_1} \sqrt{\frac{\left(\frac{\partial \gamma_x}{\partial t}\right)^2 + \left(\frac{\partial \gamma_y}{\partial t}\right)^2}{\gamma_y(t)^2}} dt \\ &\geqslant \int_{t_0}^{t_1} \frac{\left|\frac{\partial \gamma_y}{\partial t}\right|}{\gamma_y(t)} dt \\ &= \int_{\gamma(t_0)}^{\gamma(t_1)} \frac{dy}{y} \end{aligned}$$

and we conclude by summing over those intervals. In fact, because γ describes an arc of a circle, there exists $t \in [0,T]$ for which γ_y is monotone on $[0,t]$ and $[t,T]$.

2. The second inequality follows from

$$\operatorname{arcosh}(t) = \log(t + \sqrt{t^2 - 1}) \geqslant \log t$$ □

2.2 Group actions and fundamental domains

Proposition 2.11. A subgroup $\Gamma \subseteq G$ is *Fuchsian* if the following equivalent conditions hold:

1. Γ is discrete.

2. Γ acts properly discontinuously on $\mathbf{H}$.

Note that in for general continuous group actions, we only have the implication $2 \implies 1$. The nontrivial implication follows for example from the fact that $G \curvearrowright \mathbf{H}$ is a proper group action. This in turn follows from the general result (D.49) on isometry groups of metric spaces. It is possible to give an elementary proof for this particular action $\Gamma \curvearrowright \mathbf{H}$. See e.g. [Clark, 2018, Theorem 5].

Definition 2.12. A *lattice* $\Gamma \subset G$ is a Fuchsian group of finite covolume $\mathrm{vol}(\Gamma\backslash\mathbf{H})$. This is in particular the case when Γ is cocompact.

From now on, Γ will denote a lattice.

Definition 2.13. An open (closed) fundamental domain for Γ is a connected regular open (closed) set $F \subset \mathbf{H}$ for which the projection $\mathbf{H} \to \Gamma\backslash\mathbf{H}$ is:

1. injective when restricted to the interior F°,
2. surjective when restricted to the closure of $\overline{F}$,

and whose boundary ∂F is of measure zero. An open (closed) *fundamental polygon* is an open (closed) fundamental domain which is convex (for the hyperbolic metric) and whose boundary (in $P^1(\mathbb{C})$) consists of a finite number of geodesic segments together with a finite number of points in $P^1(\mathbb{R})$. In addition, we require the polygon to be *locally finite*, meaning that every compact set intersects only finitely many translates γF, $\gamma \in \Gamma$.

A convex open set is automatically regular, as is a convex closed set with nonempty interior. One can show that local finiteness of the fundamental domain is equivalent to requiring that the continuous bijjection $\overline{F}/G \twoheadrightarrow \mathbf{H}/G$ is a homeomorphism [Beardon, 1983, Theorem 9.2.4].
The precise definition of a fundamental domain is of little importance, the important property is that F with the hyperbolic measure has the same volume as the quotient $\Gamma\backslash\mathbf{H}$, thanks to the last requirement. We present two nice constructions of fundamental polygons:

Proposition 2.14. Let $w \in \mathbf{H}$ have trivial stabilizer. Then the Dirichlet polygon

$$D = \{z \in \mathbf{H} : d(z, w) < d(z, \gamma w) \quad \forall \gamma \in \Gamma - \{1\}\}$$

is an open fundamental polygon for Γ. Nontrivial fixed points for the action $\Gamma \curvearrowright P^1(\mathbb{C})$ lying on the boundary ∂D are called *vertices*, and one has that ∂D has an even number of vertices, joined by geodesic segments.

Let $A \leqslant G$ be the subgroup of diagonal matrices of determinant 1 and N be the unipotent upper-triangular group. Recall the Iwasawa decomposition

$$G = NAK \tag{2.15}$$

We call an element parabolic (hyperbolic, elliptic) if it is not the identity and conjugate to an element of N (A, K). They can be characterized in terms of their fixed points or trace:

- parabolic elements have one fixed point, which lies on $P^1(\mathbb{R})$, and trace equal to 2.
- hyperbolic elements have two fixed points, which lie on $P^1(\mathbb{R})$, and trace larger than 2.
- elliptic elements have two complex conjugate fixed points, and trace less than 2.

This gives a partition of $G - \{1\}$ into three sets. Their fixed points are called parabolic, elliptic or hyperbolic accordingly. A *cusp* for Γ is an orbit of parabolic fixed points.
One shows by direct calculation that two elements of G commute if and only if they have the same fixed points, and that they must lie in the same conjugate of N, A or K. Because discrete subgroups of N, A and K are cyclic, this shows that stabilizers are cyclic. We see that:

Proposition 2.16 (Elliptic fixed points)**.** There is a finite number of elliptic orbits under Γ. The stabilizer of an elliptic point is a finite cyclic group consisting of elliptic elements. There exist elliptic fixed points if and only if Γ has no elliptic elements, which is the case if and only if the projection $\mathbf{H} \twoheadrightarrow \Gamma\backslash\mathbf{H}$ is a covering map.

From the theory of Riemann surfaces, it follows that:

Corollary 2.17. $\Gamma\backslash\mathbf{H}$ is a Riemann surface and the projection onto it is a ramified holomorphic covering map, whose ramification points are precisely the elliptic fixed points of Γ.

Because G acts by orientation-preserving isometries, we see that $\Gamma\backslash\mathbf{H}$ is an oriented Riemannian manifold, and the metric and volume form on $\mathbf{H}$ descend to the quotient. In particular, geodesics descend to the quotient, and by Hopf–Rinow (D.39) it is still a complete Riemannian manifold.
One can show that:

Proposition 2.18 (Cusps)**.** There is a finite number of cusps for Γ. The stabilizer of a parabolic point is an infinite cyclic group consisting of parabolic elements. There are no cusps if and only of Γ has no parabolic elements, which is the case if and only if Γ is cocompact.

Let $\mathfrak{a}$ be a cusp for Γ. There exists $\sigma_{\mathfrak{a}} \in G$ with $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$, so that ∞ is a cusp for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$, and its stabilizer Γ_∞ is a cyclic group generated by a parabolic element of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad , \quad t \neq 0$$

Because

$$\begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1 & s^2 t \\ 0 & 1 \end{pmatrix}$$

we may assume $t = 1$, so that Γ_∞ is generated by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proposition 2.19. [Ford, 1929] Suppose ∞ is a cusp for Γ, with stabilizer generated by T. Let $\beta \in \mathbb{R}$ arbitrary and let $F_\infty = \{z \in \mathbf{H} : \Re\mathrm{e}\, z \in (\beta, \beta+1)\}$. Then the set

$$F = \{z \in F_\infty : |cz+d| > 1 \quad \forall \gamma \in \Gamma - \Gamma_\infty\} \tag{2.20}$$

is an open fundamental polygon. That is, it consists if the points in the vertical strip F_∞ that are exterior to the (Euclidean) half-circles with center $-d/c$ and radius $|c|^{-1}$, for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma - \Gamma_\infty$$

We will refer to such a domain (with, say, $\beta = 0$) as the *standard fundamental domain.* One shows that two translates of F share either an elliptic fixed point or one side (which may contain a fixed point), and that the transformation $\gamma \in \Gamma$ sending F to an adjacent fundamental domain, fixes the point or fixes (setwise) the side they share.

Corollary 2.21. Suppose ∞ is a cusp for Γ, with stabilizer generated by T. Then

$$c_\infty := \inf \left\{ |c| : \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma - \Gamma_\infty \right\} > 0$$

Proof. An element of the set $\Gamma_\infty\backslash\Gamma$ is determined by its bottom row. Denote it by (c,d). If $(c,d) \in \Gamma_\infty\backslash\Gamma$, then so is $(c, d+nc)$ for $n \in \mathbb{Z}$. Take any $\beta \in \mathbb{R}$, say $\beta = 0$, and consider the fundamental domain F from the previous proposition. Replacing d by $d+nc$ if necessary, we know that there is a half-disk of radius $|c|^{-1}$ centered at a point on $[0,1]$, which is disjoint from F. If c could be arbitrarily small, then F would be empty. □

If $\Gamma \subseteq \mathsf{SL}_2(\mathbb{Z})$, then trivially $c_\infty \geqslant 1$.

Corollary 2.22. Let $z = x + iy \in \mathbf{H}$ with $y > 0$ and $\gamma \in \Gamma - \Gamma_\infty$. Then, with c_∞ as above:

1. $\Im\mathrm{m}(\gamma z) \leqslant 1/(c_\infty^2 y)$.
2. $d(z, \gamma z) \geqslant 2\log(c_\infty y)$ for $c_\infty y > 1$.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. We have $\Im\mathrm{m}(\gamma z) = y/|cz+d|^2$ and $|cz + d| \geqslant |c|y \geqslant c_\infty y$ because $c \neq 0$.
2. By (2.9),
$$d(z, \gamma z) \geqslant \log \left| \frac{y}{\Im\mathrm{m}(\gamma z)} \right| \geqslant \log(c_\infty^2 y^2)$$ □

Note how the first inequality implies the following: there exists a neighborhood V of the cusp ∞, such that unless $\gamma \in \Gamma_\infty$, we have $\gamma V \cap V = \varnothing$. It is almost like saying that the action of Γ extends properly discontinuously to the cusp.

2.3 Examples

The prototypical example of a lattice is $\Gamma = \mathsf{PSL}_2(\mathbb{Z})$. If Γ has coefficients contained in a field K, then so are its cusps. Thus because the action $\mathsf{PSL}_2(\mathbb{Z}) \curvearrowright P^1(\mathbb{Q})$ is transitive, there is precisely one cusp, which we call ∞. There are two elliptic orbits: one containing i, with stabilizer of order 2, and one containing $j = \exp(2\pi i/3)$, with a stabilizer of order 3. The fundamental domain from (2.19) becomes, with $\beta = -1/2$:
$$\left\{ z = x + iy \in \mathbf{H} : |z| \geqslant 1, x \in \left[\frac{-1}{2}, \frac{1}{2}\right] \right\}$$
A subgroup Γ' of finite index in a lattice Γ is again a lattice: it has a fundamental domain which is a finite union of fundamental domains for Γ. In particular one can consider congruence subgroups of level $N \in \mathbb{N}$, they are subgroups of $\mathsf{PSL}_2(\mathbb{Z})$ which contain the kernel $\Gamma(N)$ of the surjective reduction homomorphism
$$\mathsf{PSL}_2(\mathbb{Z}) \twoheadrightarrow \mathsf{PSL}_2(\mathbb{Z}/N\mathbb{Z})$$
The construction of the lattice $\mathsf{PSL}_2(\mathbb{Z}) \subset \mathsf{PSL}_2(\mathbb{Q})$ generalizes as follows:

Definition 2.23. A lattice $\Gamma \subset \mathsf{PSL}_2(\mathbb{R})$ is arithmetic if, when $\mathcal{L}$ denotes the set of traces of elements of Γ (which are well-defined up to sign), then

(a) $\mathbb{Q}(\mathcal{L})$ is a finite extension of $\mathbb{Q}$ and $\mathcal{L} \subseteq \mathcal{O}_K$.

(b) If $\phi : K \to \mathbb{C}$ is an embedding such that $\phi(\mathcal{L})$ is unbounded, then $\phi(t) = \pm t$ for all $t \in \mathcal{L}$.

All arithmetic lattices can be constructed out of quaternion algebras over number fields. The above characterization is due to Takeuchi [Takeuchi, 1975]. In particular, the set of arithmetic lattices is countable, while the set of all lattices is uncountable.

2.4 Fourier expansions

Let $\Gamma \subset G$ be a lattice, with a cusp at ∞ and with stabilizer Γ_∞ generated by T. Let $f : \mathbf{H} \to \mathbb{C}$ be a smooth Γ-invariant function. Then in particular $f(x + iy)$ is Γ_∞-invariant for fixed y, and we can Fourier-expand it:

Proposition 2.24. There exist functions $\widehat{f}_n(y)$ for $n \in \mathbb{Z}$, with

$$f(z) = \sum_{n\in\mathbb{Z}} \widehat{f}_n(y)e(xn) \tag{2.25}$$

where we denote $e(u) = \exp(2\pi iu)$ for brevity. They are given by

$$\widehat{f}_n(y) = \int_0^1 f(x+iy)e(-nx)dx \tag{2.26}$$

We call $\widehat{f}_0(y)$ the constant term, and we denote it also by C_f. Perhaps misleadingly, it is not a constant function.
When Γ has a cusp $\mathfrak{a} \neq \infty$, we can give a similar Fourier expansion: take $\sigma_\mathfrak{a} \in G$ with $\sigma_\mathfrak{a}\infty = \mathfrak{a}$, so that ∞ is a cusp for $\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}$. The Fourier expansion at ∞ w.r.t. this lattice will serve as the Fourier expansion at the cusp $\mathfrak{a}$. In order to avoid using heavy notation, we will almost always assume that Γ has a cusp at ∞.
We see from (2.26) that the $\widehat{f}_n(y)$ are smooth. Now suppose that f is an eigenfunction of the Laplacian $-\Delta$ with eigenvalue λ. By Fourier expanding f, we have separated the variables x and y, so that we expect to obtain differential equations in the $\widehat{f}_n(y)$. Indeed, with the formula for $-\Delta$ (D.46), we find:

Proposition 2.27. For all $n \in \mathbb{Z}$, $\widehat{f}_n(y)$ is a solution of the differential equation

$$y^2F'' + (\lambda - 4\pi^2n^2y^2)F = 0$$

Proof. By repeated partial integration, we have that the Fourier coefficients are rapidly decreasing, in a locally uniform way when y varies:

$$|\widehat{f}_n(y)| \leqslant |2\pi n|^{-p} \int_0^1 |f^{(p)}(x+iy)|dx \qquad \forall p > 0, n \neq 0$$

and similarly for the derivatives $\widehat{f}_n^{(k)}(y)$. We may thus apply $-\Delta$ termwise to the Fourier expansion, and we obtain

$$-\Delta f(z) = -\sum_{n\in\mathbb{Z}} y^2(\widehat{f}_n^{(2)}(y) - 4\pi^2n^2\widehat{f}_n(y))e(xn)$$

The claim follows by comparing the Fourier coefficients (which are unique) with those of

$$\lambda f(z) = \sum_{n\in\mathbb{Z}} \lambda\widehat{f}_n(y)e(xn)$$ □

For the constant term, we find:

Proposition 2.28. Let $\lambda = s(1-s)$ with $s \in \mathbb{C}$. Then the constant term C_f is a linear combination of

$$\begin{cases} y^s \text{ and } y^{1-s} & : s \neq \frac{1}{2} \\ y^s \text{ and } y^s \log y & : s = \frac{1}{2} \end{cases}$$

Proof. By inspection, those are linearly independent solutions to the differential equation $F'' = -\lambda y^{-2}F$. □

Now consider $n \neq 0$. Substituting $u = 2\pi|n|y$ gives the following equation for $G(u) = \widehat{f}_n(u/(2\pi|n|))$:

$$G''(u) + (\lambda u^{-2} - 1)G(u) = 0$$

This differential equation is studied in (H): we have $G(u) = W(2u)$ for some Witthaker function W. More precisely, let $\lambda = s(1-s)$ so that $\lambda = \frac{1}{4} - m^2$ with $m = \pm\left(s - \frac{1}{2}\right)$. Suppose $\Re\mathrm{e}\, s > 0$. Then

$$\widehat{f}_n(y) = c_1W_{0,s-1/2}(4\pi|n|y) + c_2W_{0,s-1/2}(-4\pi|n|y)$$

is a linear combination of the Witthaker functions from (H.6). Using a more careful analysis of Whittaker functions, one can generalize this to all s, with the condition $\Re\mathrm{e}\, s > 0$.

Proposition 2.29. Suppose in addition that $f(z) = o(e^{2\pi y})$ as $y \to \infty$, uniformly in x. Then the same asymptotic holds for the $\widehat{f}_n(y)$, and we have

$$\widehat{f}_n(y) = a_n W_{0,s-1/2}(4\pi|n|y)$$

for some $a_n \in \mathbb{C}$.

Proof. That the $\widehat{f}_n(y)$ satisfy the same asymptotic relation, follows from their definition (2.26). Now, by (H.6), the Whittaker function $W_{0,s-1/2}(-4\pi|n|y)$ is asymptotically equivalent to $e^{2\pi|n|y}$ as $y \to \infty$, hence $\widehat{f}_n(y)$ is a linear combination of $W_{0,s-1/2}(4\pi|n|y)$ alone. □

Corollary 2.30. When $f(z) = o(e^{2\pi y})$ as $y \to \infty$, uniformly in x, is an eigenfunction of $-\Delta$ with eigenvalue $s(1-s)$, then $f(z) - C_f = O(e^{-2\pi y})$ as $y \to \infty$, uniformly in x.

Proof. We have $\widehat{f}_n(y) = O_n(e^{-2\pi|n|y})$ for all $n \neq 0$, because it is a scalar multiple of a decaying Whittaker function. We want to get rid of the dependence on n, by estimating the coefficients a_n. Recall that

$$|\widehat{f}_n(y)| \leqslant \int_0^1 |f(x+iy)|dx$$

Evaluating in small y gives

$$|a_n| \ll_\epsilon e^{\epsilon|n|}$$

so that $f(z) - C_f$ is bounded by the geometric series

$$\sum_{n\neq 0} e^{\epsilon|n|} e^{-2\pi|n|y} \ll e^{\epsilon - 2\pi y} \quad , \quad \epsilon - 2\pi y < 0$$

We conclude by noting that any $\epsilon < 2\pi$ works for all $y \geqslant 1$. □

3 Operators on symmetric spaces

When M is a smooth manifold, one wants to understand linear operators on $C^\infty(M)$. Differential operators in particular, but not unrelated is the notion of convolution operators: It is a particularly useful tool because it gives a way to regularize non-smooth functions, and because of the existence of *approximations of the identity*: Define a compactly supported smooth function ρ on $\mathbb{R}^n$ by

$$\rho(x) = c \cdot \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & : |x| < 1 \\ 0 & : x \geqslant 1 \end{cases} \tag{3.1}$$

where $c > 0$ is used to normalize the function so that $\int_{\mathbb{R}^n} \rho = 1$. Define

$$\rho_\delta(x) = \delta^{-n}\rho(x/\delta) \tag{3.2}$$

We then have for all $\delta > 0$ that $\int_{\mathbb{R}^n} \rho_\delta = 1$. For $f \in L^1(\mathbb{R}^n)$ resp. $f \in C(\mathbb{R}^n, \mathbb{C})$ the convolution $\rho_\delta * f$ is defined and

$$\rho_\delta * f \to f$$

for L^1-convergence resp. locally uniform convergence. From now on let $f, g \in C^\infty(\mathbb{R}^n)$ be smooth and observe that the identity

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

(valid whenever any of the integrals converges) does not only show commutativity of the convolution, but also tells us something about the action of translation-invariant differential operators (as defined in (F.27)): Let $D = D^\alpha$ be a monomial in the $\partial/\partial x_i$ (for the notation, see (F.1)), then

$$D(f * g) = \int_{\mathbb{R}^n} (Df)(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)(Dg)(x-y)dy$$

That is:

$$D(f * g) = (Df) * g = f * (Dg) \tag{3.3}$$

whenever we can switch the order of integration and differentiation, which is for example the case when either f or g has compact support or when they are both in the *Schwartz space*

$$S(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : |D^\alpha f(x)| \ll |x|^{-M} \quad \forall \alpha \in \mathbb{N}^n, M > 0 \right\}$$

The existence of approximations of the identity then allows an unusual proof of the following

Proposition 3.4. Let $f : U \to \mathbb{R}$ have continuous mixed partial derivatives of order 2 on an open set $U \subseteq \mathbb{R}^2$. Then

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

on U.

Proof. The usual proof presented in a multivariable calculus course goes along the following lines: Fix $(a, b) \in U$, let $I_\epsilon = [a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon]$ and use Fubini's theorem to choose the order of integration in

$$\iint_{I_\epsilon} \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_2 dx_1 \text{ and } \iint_{I_\epsilon} \frac{\partial^2 f}{\partial x_2 \partial x_1} dx_1 dx_2$$

Use the fundamental theorem of calculus to show that the integrals are equal. Then use a mean value theorem to conclude that, as $\epsilon \to 0$:

$$(2\epsilon)^2 \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}(a, b) + o(1) \right) = (2\epsilon)^2 \left(\frac{\partial^2 f}{\partial x_2 \partial x_1}(a, b) + o(1) \right)$$

Divide by $(2\epsilon)^2$ and let $\epsilon \to 0$. □

We present here two other proofs.

First proof. Write $D_1 = \partial/\partial x_1$ and $D_2 = \partial/\partial x_2$. Then

$$D_1 D_2(\rho_\delta * f) = D_1(\rho_\delta * D_2 f) = \rho_\delta * D_1 D_2 f$$

on the one hand and

$$D_1 D_2(\rho_\delta * f) = D_1(D_2\rho_\delta * f) = D_2\rho_\delta * D_1 f = \rho_\delta * D_2 D_1 f$$

on the other. Taking the point-wise limit for $\delta \to 0$ gives the result! □

Second proof. We can alter the argument slightly as follows: By commutativity of convolution:

$$D_1\rho_{\delta_1} * D_2\rho_{\delta_2} * f = D_2\rho_{\delta_2} * D_1\rho_{\delta_1} * f \tag{3.5}$$

for $\delta_1, \delta_2 > 0$. We want to let $\delta_1, \delta_2 \to 0$. This is tricky. We may assume that f has compact support, without changing f on a sufficiently small open set, by multiplying it with a bump function. Then taking limits is allowed:
Fix δ_2 and let $\delta_1 \to 0$. The LHS converges (pointwise) to $D_1(D_2\rho_{\delta_2} * f)$, there is no problem. The RHS is a problem of interchanging limit and integral. For the RHS, we have that $D_1\rho_{\delta_1} * f \to D_1 f$ locally uniformly, and thus uniformly because its support is contained in $\overline{B}(\operatorname{supp} f, 1)$. In particular, it is uniformly bounded as $\delta_1 \to 0$, and we can apply dominated convergence to justify the pointwise convergence

$$D_2\rho_{\delta_2} * (D_1\rho_{\delta_1} * f) \to D_2\rho_{\delta_2} * D_1 f \quad (\delta_1 \to 0)$$

where $D_2\rho_{\delta_2} \in L^1(\mathbb{R}^n)$ is used as a majorant. We obtain

$$D_1(D_2\rho_{\delta_2} * f) = D_2\rho_{\delta_2} * D_1 f$$

We could also have obtained this by using that convolution is continuous as a map $L^1 \times L^1 \to L^1$ (by Young's inequality).
We now want to let $\delta_2 \to 0$. The convergence of the RHS is no problem. For the LHS, the problem is to interchange limit and differentiation. We see from the RHS that this converges locally uniformly as $\delta_2 \to 0$. We also have that $D_2\rho_{\delta_2} * f$ converges locally uniformly, in particular, at at least one point. This means that we interchange limit and D_1 in the LHS :

$$D_1(D_2\rho_{\delta_2} * f) \to D_1(D_2 f) \quad (\delta_2 \to 0)$$

pointwise, and in fact locally uniformly. □

While a lot less elementary (we use the dominated convergence theorem when interchanging integration and differentiation), the two new proofs provide a framework for possible generalizations. The difference between them is subtle. The first proof relied on the nature of the action of differentiation as summarized in (3.3). In the second proof we only used the observation that $D\rho_\delta * f = \rho_\delta * Df$, and instead of exploiting this as in the first proof, we used the fact that convolution is commutative.

3.1 Weakly symmetric spaces

One can hope to generalize these powerful tools to Riemannian manifolds other than $\mathbb{R}^n$, to study commutativity of differential operators invariant under some group of isometries. The difficulty lies in finding an analogue of convolution, satisfying a property similar to (3.3). When G is a Lie group with (say) left-invariant Haar measure μ and $f, g \in C^\infty(G, \mathbb{C})$ such that at least one of them has compact support, we can define

$$(f * g)(x) = \int_G f(xy^{-1})g(y)d\mu(y)$$

When D is a differential operator that is invariant under right translations, we have that $D(f * g) = (Df) * g$. But unless G is abelian, we do not have $f * g = g * f$. We take a second look at how (3.3) is derived, for a differential operator D on $\mathbb{R}^n$. One way is to make a substitution in the integral, but we don't expect to have access to that in general. We try to forget the fact that $x-y$ is a subtraction, and look at its symmetries instead. Let $k(x,y)=f(x-y):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$. It is almost symmetric: when $\mu:x\mapsto -x$ denotes the inversion, then $k(x,y)=k(\mu y,\mu x)$, and we have, denoting the argument on which D acts with a subscript,

$$\begin{aligned}(D_xk)(x_0,y_0)&=(D_xk(\mu y,\mu x))(x_0,y_0)\\&=(D_xk(\mu y_0,\mu x))(x_0)\\&=((\mu_*D)_yk(\mu y_0,y))(\mu x_0)\\&=(\mu_*D)_yk(\mu y_0,\mu x_0)\end{aligned}$$

Where μ_*D is the pushforward of D by μ, which is the differential operator defined by

$$(\mu_*Df)(x_0)=D(f(\mu x))(\mu^{-1}x_0)$$

as in Appendix F. Now of course, in this case $D_xk(x_0,y_0)$ is just $D_xk(\mu y_0,\mu x_0)$, but we look for a conceptual way to understand this. Let G be the translation group on $\mathbb{R}^n$. Note that k is invariant for the under the diagonal action $G\curvearrowright\mathbb{R}^n\times\mathbb{R}^n$ on point pairs:

Definition 3.6. Let S be a Riemannian homogeneous G-space. A *point-pair invariant* on S is a smooth function $k:S\times S\to\mathbb{C}$ with

$$k(\sigma x,\sigma y)=k(x,y)\quad,\quad\forall\sigma\in G$$

Because D_x is G-invariant by assumption, D_xk is again a point-pair invariant. Note also that for all $x_0,y_0\in\mathbb{R}^n$, there exists $\sigma\in G$ such that both $\sigma x_0=\mu y_0$ and $\sigma y_0=\mu x_0$: it suffices to take σ to be the translation by $-x_0-y_0$. We conclude that

$$D_xk(x_0,y_0)=D_xk(\sigma x_0,\sigma y_0)=D_xk(\mu y_0,\mu x_0)$$

and consequently,

$$(D_xk)(x_0,y_0)=((\mu^*D)_yk(x,y))(x_0,y_0)$$

We have managed to shift the operator from the first to the second argument, by replacing it with its pullback. Now let $D^{(1)}$ and $D^{(2)}$ be G-invariant differential operators on $\mathbb{R}^n$ and note that:

$$\begin{aligned}D_x^{(1)}D_x^{(2)}k&=D_x^{(1)}(\mu^*D^{(2)})_yk\\&=(\mu^*D^{(2)})_yD_x^{(1)}k\\&=D_x^{(2)}D_x^{(1)}k\end{aligned}$$

where in the last step we used that $D_x^{(1)}k$ is again a point-pair invariant. That is, the action of translation-invariant differential operators on translation-point-pair invariants is commutative.[1]
We are not done yet. We still need a notion of approximations of the identity, and an analogue for the identity $Df*g=f*Dg$. The first ingredient is not a problem:

Definition 3.7. Let S be a homogeneous Riemannian manifold with distance function d. By homogeneity, there exists $\epsilon>0$ such that $B(x,\epsilon)$ is a normal neighborhood of x for each $x\in S$. By (D.37), the function $d(x,y)^2$ is smooth on the set of point-pairs that are at distance less than ϵ. For $\delta<\epsilon$, define now the smooth function, analogous to (3.2):

$$\rho_\delta(x,y)=c_\delta\cdot\begin{cases}\exp\left(-\dfrac{1}{1-\left(\frac{d(x,y)}{\delta}\right)^2}\right) & :d(x,y)<\delta\\0 & :d(x,y)\geqslant\delta\end{cases}\tag{3.8}$$

where c_δ is chosen to have $\int_S\rho_\delta(x,y)dy=1$ for all x.

[1]It is important to note that, in fact, we are cheating here: when switching the order of differentiation, we are using the fact that the differential operators D_x and D'_y on $\mathbb{R}^n\times\mathbb{R}^n$ commute, for given differential operators D,D' on $\mathbb{R}^n$, while that's precisely what we're trying to show! That is, this proof is circular. But it gives a way to reduce the statement about more general manifolds, to the case of $\mathbb{R}^n$.

Proposition 3.9. For $f \in C(S, \mathbb{C})$, we have

$$\rho_\delta \star f \to f$$

locally uniformly.

Proof. The proof is the same as for convolution on $\mathbb{R}^n$. Write

$$\begin{aligned}\rho_\delta \star f - f &= \int_S \rho_\delta(x,y)(f(y) - f(x))dy \\ &\leqslant \sup_{y \in B(x,\delta)} |f(y) - f(x)|\end{aligned}$$

and we conclude using Heine's theorem, which says that f is locally uniformly continuous. □

The previous discussion gives rise to a notion of weakly symmetric spaces:

Definition 3.10. [Selberg, 1956] A weakly symmetric (Riemannian) space is a triple (S, G, μ) with S is a Riemannian manifold, G a locally compact transitive group of isometries of S and μ an isometry (not necessarily in G) such that:

1. $\mu^2 \in G$
2. $\mu G \mu^{-1} = G$
3. for all $x, y \in S$ there exists $\sigma \in G$ with

$$\sigma x = \mu y \quad , \quad \sigma y = \mu x$$

The prime example is $\mathbb{R}^n$ with G the translation group and $\mu : x \mapsto -x$ the reflection.
This gives rise to a large class of Riemannian manifolds on which the algebra of invariant differential operators is commutative. Because we won't need those general results, and in order to avoid technical difficulties, we will limit our attention to the smaller class of symmetric spaces (E.9) without attempting to state the results under minimal hypotheses.

3.2 Point-pair invariants

Proposition 3.11 (Invariant differential operators applied to point-pair invariants)**.** Let S be a symmetric space.

1. Applying an invariant differential operator $D \in \mathscr{D}(S)$ to either argument of k yields again a point-pair invariant.
2. We have an unambiguous action of $\mathscr{D}(S)$ on point-pair invariants: if we denote the action of D on the first argument by D_x and on the second by D_y, then

$$D_x k(x_0, y_0) = D_y k(x_0, y_0)$$

3. The action of $\mathscr{D}(S)$ on point-pair invariants is commutative: for invariant differential operators $D_1, D_2 \in \mathscr{D}(S)$ we have:

$$D_1 D_2 k = D_2 D_1 k$$

Proof. 1. We have

$$D_y k(x_0, y_0) = D_y(k(\sigma x_0, \sigma y))|_{y=y_0} = (D_y k)(\sigma x_0, y)|_{y=\sigma y_0}$$

for $\sigma \in G$, where the last equality follows from the invariance of D. Similarly for the first argument.

2. Because k is symmetric, we have

$$D_x k(x_0, y_0) = D_x(k(y, x))|_{x=x_0, y=y_0} = (D_y k)(y_0, x_0)$$

where the last equality follows from the chain rule. Because $D_y k$ is symmetric, this equals $D_y k(x_0, y_0)$.

3. Follows from

$$D_1 D_2 k = D_{1,x} D_{2,y} k = D_{2,y} D_{1,x} k = D_2 D_1 k$$ □

Definition 3.12 (Point-pair invariant of compact support)**.** Let S be a symmetric space with isometry group G. A point pair-invariant k on S has *compact support* if the following equivalent conditions hold:

1. There exists $y_0 \in S$ such that $k(\cdot, y_0)$ has compact support.
2. For all $y \in S$, the function $k(\cdot, y)$ has compact support.
3. For all $y_0 \in S$, there exists a neighborhood U of y_0 and a compact $T \subseteq S$ such that the support of $k(\cdot, y)$ is contained in T for all $y \in U$.
4. For compact $V \subseteq S$, the restriction $k : V \times S \to \mathbb{R}$ has compact support.

We denote by $A(S)$ the set of point-pair invariants of compact support.

Proof of equivalence. $1 \implies 2$: Let $\sigma \in G$ with $\sigma y_0 = y$, then

$$\operatorname{supp} k(\cdot, y) = \sigma \cdot \operatorname{supp} k(\cdot, y_0)$$

is compact.

$2 \implies 3$: By (E.8) with $x_0 = y_0$ there exists a compact neighborhood U of y_0 and a homeomorphism on its image $\psi : U \to G$ such that $\psi(y)y_0 = y$ for all $y \in U$. Then

$$\operatorname{supp} k(\cdot, y) = \psi(y) \operatorname{supp} k(\cdot, y_0) \subseteq \psi(U) \cdot \operatorname{supp} k(\cdot, y_0)$$

which is compact, because $\psi(U)$ and $\operatorname{supp} k(\cdot, y_0)$ are.

$3 \implies 4$: We can cover V by a finite number of relatively compact open sets U_i such that the restriction of k to $U_i \times S$ has support contained in a compact set $\overline{U}_i \times T_i$. Then the restriction to $V \times S$ has support contained in the compact set $\bigcup \overline{U}_i \times T_i$.

$4 \implies 1$: Take $V = \{y_0\}$. □

Remark 3.13. In general, for a smooth function $k : M \times M \to \mathbb{R}$ these conditions are not equivalent. That is, compact support of $k(y, \cdot)$ at each point y does not imply that locally in y the support lies in one uniform compact set. Consider for example $M = \mathbb{R}$ and $k : \mathbb{R} \times \mathbb{R}$ which has smooth bumps on each of the rectangles

$$\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \times [2n, 2n+1] \qquad (n \geqslant 0)$$

and is 0 elsewhere. Then all $k(y, \cdot)$ have compact support, $k(0, \cdot)$ has empty support, but the restriction of k to $[-\delta, \delta] \times \mathbb{R}$ has unbounded support for all $\delta > 0$.

Definition 3.14. Consider the set $A(S)$ of point-pair invariants of compact support. We equip it with pointwise addition and the multiplication

$$(k_1 \circ k_2)(x, y) = \int_S k_1(x, w) k_2(w, y) dw$$

This makes $A(S)$ a commutative algebra:

Proposition 3.15. The composition $k_1 \circ k_2$

1. is smooth.
2. is a point-pair invariant.
3. has compact support.
4. satisfies $k_1 \circ k_2 = k_2 \circ k_1$.

Proof. 1. Because k_1, k_2 have compact support, locally in x and y the integrand has compact support, hence defines a smooth function.

2. Because G acts by isometries:

$$\begin{aligned}\int_S k_1(\sigma x, w)k_2(w, \sigma y)dw &= \int_S k_1(x, \sigma^{-1}w)k_2(\sigma^{-1}w, y)dw \\ &= \int_S k_1(x, w)k_2(w, y)dw\end{aligned}$$

3. Fix x, then the support of $k_1(x, \cdot)$ is compact, call it T. Because T is compact, the support of $k_2 : S \times T \to \mathbb{R}$ (as a function, not as a point-pair invariant) is also compact, hence the integrand is identically 0 for y outside of a compact set.

4. Because point-pair invariants are symmetric,

$$\int_S k_1(x, w)k_2(w, y)dw = \int_S k_1(w, x)k_2(y, w)dw$$ □

Many results about composition of point-pair invariants carry through if at least one of them has compact support. We will at times use such results without explicitly mentioning that they the conditions are not strictly satisfied because of issues with supports.

3.3 Radially symmetric functions

Let S be a symmetric space with isometry group G. We have a right group action of G on real or complex-valued functions by $L_g f(x) = f(gx)$.[2]

Definition 3.16. Let S be a symmetric space with isometry group G, let $x_0 \in S$ and $f \in C^\infty(S)$ smooth. Then f is *radially symmetric* about x_0 if it is invariant under the stabilizer of x_0 in G.

Example 3.17. If k is a point-pair invariant, then $k(\cdot, x_0)$ is radially symmetric about x_0.

We can make every function on a symmetric space radially invariant about a point, as follows. The stabilizer K of x_0 is a compact Lie group by (D.50). It has a unique right-invariant Haar measure μ such that K has volume 1. (And because compact groups are unimodular, it is in fact bi-invariant.)

Definition 3.18. The *symmetrization* of f about x_0 is

$$f^{\text{rad}}_{x_0}(x) = f(x, x_0) = \int_K f(rx)\mu(dr)$$

Proposition 3.19 (Properties of symmetrization)**.** Let M be a Riemannian manifold, $x_0 \in M$ with stabilizer K, and $f \mapsto f^{\text{rad}}_{x_0}$ denote the symmetrization map.

1. $f(x, x_0)$ is radially symmetric.
2. f is radially symmetric if and only if $f(x) = f(x, x_0)$.
3. $f(x_0, x_0) = f(x_0)$.

[2]Sometimes denoted R_g.

4. If f is smooth, so is $f(x, x_0)$.
5. If D is a K-invariant differential operator, then $(Df)(\cdot) = D(f(\cdot, x_0))$.

Proof. 1. It is radially symmetric because for $g \in K$:

$$f(gx, x_0) = \int_K f(rgx)\mu(dr) = \int_K f(rx)(R_g)_*\mu(dr) = \int_K f(rx)\mu(dr)$$

because μ is right-invariant, where $R_g : h \mapsto hg$ denotes the right regular representation of $g \in K$.

2, 3. Because $\int_K d\mu = 1$.

4, 5. Because K is compact and f smooth, we can switch integration and differentiation. □

There is a converse for (3.17). Each radially symmetric $k(x, x_0)$ can be extended to a point-pair invariant on $S \times S$, as follows. Take $(x, y) \in S \times S$, and let $\sigma \in G$ such that $\sigma y = x_0$. Define

$$k(x, y) := k(\sigma x, x_0)$$

Proposition 3.20. *k is*

1. *well-defined, i.e. does not depend on the choice of σ.*
2. *a point-pair invariant.*
3. *smooth.*

Proof. 1. It is well-defined precisely because we assume that $k(x, x_0)$ is radially symmetric.

2. This follows from the construction and well-definedness: let $x, y \in S$, $\sigma y = x_0$ and $\tau \in G$. Then $(\sigma\tau^{-1})(\tau y) = x_0$ and hence:

$$k(\tau x, \tau y) := k(\sigma\tau^{-1}\tau x, x_0) = k(\sigma x, x_0) = k(x, y)$$

3. By (E.8), we can choose σ in a smooth way as a function of y, in a neighborhood of any y_0. □

To summarize, we have the following correspondence:

Theorem 3.21. *For each point $x_0 \in S$ there is a bijection between radially invariant functions $g(x)$ and point-pair invariants $k(x, y)$. Under this bijection:*

1. *$g(x) = k(x, x_0)$.*
2. *Dg corresponds to Dk for $D \in \mathscr{D}(S)$, the action of D on k being unambiguous by (3.11).*

3.4 Isotropic spaces

In the case of isotropic Riemannian manifolds, such as $\mathbf{H}$, radially symmetric functions are easiest to understand:

Proposition 3.22. *Let S be an isotropic symmetric space. Then there exists $\delta > 0$ such that:*

1. *Every radially symmetric function f around $x_0 \in S$ is locally a function of the radial distance r to x_0: there exists $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $f = g \circ r$ in the geodesic ball $B(x_0, \delta)$. Moreover, g is smooth on $[0, \delta)$.*
2. *For every point-pair invariant k there exists a smooth $g : [0, \delta) \to \mathbb{R}$ such that for all $x, y \in S$ with $0 < d(x, y) < \delta$ we have $k(x, y) = g(d(x, y))$. Moreover, g is smooth on $[0, \delta)$.*

Proof. By (E.2), G acts transitively on sufficiently small geodesic spheres around x_0, say those of radius $< \delta$.

1. Thus f is locally a function of r, on $B(x_0,\delta)$. We want that g is smooth. Note that $r : B(x_0,\delta) \to [0,\delta)$ has a smooth section: Take a normal coordinate chart $\phi : B(x_0,\delta) \to \mathbb{R}^n$ (§D.4.1) and define $h(r) = \phi^{-1}(r, 0, \ldots, 0)$. Then $g = f \circ h$ is smooth.

2. For fixed x, the function $k(x, \cdot)$ is radially symmetric, and the above applies. □

Remark 3.23. 1. One can show that g is a smooth function of r^2 resp. $d(x,y)^2$. This follows from the fact that a smooth even function $f(x)$ on $\mathbb{R}$ is a smooth function of x^2 (3.24).

2. In the case of $\mathbf{H}$, every two points are joined by a unique geodesic, hence by (D.41) the exponential maps are global diffeomorphisms, and we can take $\delta = \infty$ in the above proposition.

Proposition 3.24. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth and even function. Then there exists a smooth $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $f(x) = g(x^2)$.

Proof. See the Mathoverflow post [Mathoverflow, 2011] for various proofs. □

Proposition 3.25. Let M be a Riemannian manifold, $x_0 \in M$, (x^i) normal coordinates at x_0 and r the radial distance to x_0, defined on an open neighborhood U of x_0. Let $f : U - \{x_0\} \to \mathbb{R}$ be smooth with $f = h \circ r$ for some $h : \mathbb{R}_{>0} \to \mathbb{R}$. Then

$$-\Delta f = D(h) \circ r$$

for some differential operator D of degree 2, whose highest degree coefficient is nonzero for $t \in \mathbb{R}_{>0}$ sufficiently small.

Proof. Note that h is automatically smooth by (3.22). That such D of degree 2 exists follows immediately from (D.45) and the chain rule. Let's look at the highest degree coefficient. If g is the metric and g^{ij} the components of the inverse of its matrix in the coordinates (x^i), then the highest degree coefficient is

$$\frac{1}{r^2}\sum_{i,j} g^{ij} x^i x^j$$

Because $g^{ij} = \delta_{ij}$ at x_0, by continuity this is close to $r^{-2}\sum_i (x^i)^2 = 1$ in a neighborhood of x_0, hence nonzero. □

3.5 Integral operators

On any measure space X we can consider integral operators on $L^2(X)$ of the form

$$f \mapsto \int_X k(z,w) f(w) dw$$

where $k \in L^2(X \times X)$ is a *kernel*. See (A.7). We will denote the action on f by $k \star f$ and the operator itself simply by k. If $X = S$ is a symmetric space, k_1, k_2 are compactly supported point-pair invariants and $k_1 \circ k_2$ their composition from (3.14), then by Fubini:

$$(k_1 \circ k_2) \star f = k_1 \star (k_2 \star f)$$

Integral operators whose kernel is a point-pair, are related to radially symmetric functions as follows:

Proposition 3.26. Let S be a symmetric space, $x_0 \in S$, g radially symmetric about x_0, $f \in C^\infty(S)$ and $k \in A(S)$ a compactly supported point-pair invariant. Let h be the point-pair invariant extension of g from (3.21). Then:

1. Convolution commutes with the bijection between point-pair invariants and radially symmetric functions: $k \circ h$ is the point-pair invariant associated to $k \star g$.

2. Convolution commutes with symmetrization:

$$(k \star f)^{\mathrm{rad}}_{x_0} = k \star f^{\mathrm{rad}}_{x_0}$$

Proof. 1. Because

$$(k \circ h)(x, x_0) = \int_S k(x, y)h(y, x_0)dy = \int_S k(x, y)g(y)dy = (k \star g)(x)$$

we have that $k \circ h$ is necessarily the unique extension of $k \star g$.

2. Let R be the isotropy subgroup of x_0, then the RHS evaluated in x is

$$\begin{aligned}\int_S k(x, y) \int_R f(ry)drdy &= \int_R \int_S k(x, y)f(ry)dydr \\ &= \int_R \int_S k(x, r^{-1}y)f(y)dydr \\ &= \int_R \int_S k(rx, y)f(y)dydr\end{aligned}$$

where we were allowed to apply Fubini because by assumption on k and compactness of R, the integrands have compact support. □

Proposition 3.27 (Convolution and invariant differential operators)**.** For $k \in A(S)$ compactly supported, $D \in \mathscr{D}(S)$ invariant and $f \in C^\infty(S)$:

$$(Dk) \star f = D(k \star f) = k \star Df$$

Proof. The first equality is immediate, because we can differentiate under the integral sign:

$$\int_S D_x k(x, y)f(y)dy = D_x \int_S k(x, y)f(y)dy$$

For the second equality, fix $x_0 \in S$. We may suppose f is radially symmetric about $x_0 \in S$, because D and $k\star$ commute with symmetrization ((3.19).5) and (3.26).2) and we haven't changed the value of any of the involved functions at x_0.
Under the bijection from (3.21), Say f corresponds to h. Then we have the correspondences:

$$\begin{aligned}k \star Df &\leftrightarrow k \circ Dh \\ Df &\leftrightarrow Dh \\ f &\leftrightarrow h \\ k \star f &\leftrightarrow k \circ h \\ D(k \star f) &\leftrightarrow D(k \circ h)\end{aligned}$$

by (3.26)(1) and (3.21). Because $D(k \circ h) = k \circ Dh$, it follows that $k \star Df = D(k \star f)$. □

3.6 The algebra of invariant differential operators

We continue the discussion in the very beginning of this section. For real Lie groups G, we showed that the algebra of left-invariant differential operators is isomorphic to its universal enveloping algebra $U(\mathfrak{g})$ (F.39) with the Poincaré–Birkhoff–Witt theorem as a corollary. One can show that the algebra of bi-invariant differential operators is isomorphic to the center of $U(\mathfrak{g})$. In particular, it is commutative. The same conclusion holds for symmetric spaces. We give two proofs that are very different from the proof for Lie groups. They mimic the two proofs of the analogous result for $\mathbb{R}^n$ (3.4).

Theorem 3.28. Let S be a symmetric space. Then the algebra of invariant differential operators $\mathscr{D}(S)$ is commutative.

First proof. [Selberg, 1956] Let f be a smooth function and $x_0 \in S$. By (3.19) and (3.21) there exists a point-pair invariant k such that $Dk(x_0, x_0) = Df(x_0)$ for all $D \in \mathscr{D}(S)$. By (3.11), we have $D_1 D_2 f(x_0) = D_2 D_1 f(x_0)$ for all $D_1, D_2 \in \mathscr{D}(S)$. □

The ingredients for the proof were: homogeneity of S, the compactness of the isotropy groups of one (hence every) point of S, and the fact that every two points are switched (i.e. that point-pair invariants are symmetric.)
It is stable by the action of $\mathscr{D}(S)$ on point-pair invariants by differentiation with respect to the first variable. We will assume that the action is on the first variable in the proof below.

Second proof of (3.28). For small $\delta > 0$ we have a compactly supported point-pair invariant $\rho_\delta(x, y)$ that is an 'approximation of the identity', and for smooth f:

$$\int_S \rho_\delta(x, y) f(y) dy \to f(x) \text{ as } \delta \to 0$$

where the convergence is locally uniformly (3.9). Take invariant $D_1, D_2 \in \mathscr{D}(S)$ and small $\delta_1, \delta_2 > 0$. Then $D_1\rho_{\delta_1}$ and $D_2\rho_{\delta_2}$ commute. We have:

$$\int_S (D_1\rho_{\delta_1} \circ D_2\rho_{\delta_2})(x, y) f(y) dy = \int_S D_1\rho_{\delta_1}(x, w) \int_S D_2\rho_{\delta_2}(w, y) f(y) dy dw$$

by Fubini, because for fixed x the integrands have compact support. It remains to show that we can interchange limits, differentiation and integration to show that this approaches $(D_1D_2f)(x)$ as $\delta_1, \delta_2 \to 0$. By symmetry, we then have $(D_1D_2f)(x) = (D_2D_1f)(x)$. The validity of this interchanging can be checked in the exact same way as we did in the second proof for differential operators on $\mathbb{R}^n$ (3.4). □

With more representation theory, it is possible to give a proof of the same flavor as the proof for Lie groups. Because S is homogeneous, we can write it as a quotient of a Lie group G by a compact Lie group K. The *Harish-Chandra homomorphism* provides a map from a commutative algebra constructed from the Lie algebras of $\mathfrak{g}$ and $\mathfrak{h}$, to the algebra $\mathscr{D}(G/K)$ of invariant differential operators on S. See e.g. [Helgason, 1984, Theorem 5.13].

3.7 Selberg's eigenfunction principle

Point-pair invariant integral operators have another very useful property. Recall that they respect radial symmetry (3.26), preserve smoothness and commute with invariant differential operators (3.27). The idea of the Selberg eigenfunction principle is to exploit these properties by letting point-pair invariants act on a vector space of dimension 1.

Example 3.29. On $\mathbf{H}^{n+1}$, for $s \in \mathbb{C}$, the functions y^s and y^{n-s} are eigenfunctions for the Laplacian with eigenvalue $s(n - s)$.

Proof. From the formula from (D.46) for $-\Delta$. □

Proposition 3.30 (Spherical eigenfunctions)**.** Let S be an isotropic Riemannian manifold, $z_0 \in S$ and $\lambda \in \mathbb{C}$.

1. There exists a punctured open neighborhood V of z_0 such that every radially symmetric eigenfunction for $-\Delta$ defined on some open subset of V extends globally to V in a unique way, and the kernel $\ker(\Delta + \lambda)$ has complex dimension exactly 2 on V.

2. Let $S = \mathbf{H}$ and $\lambda = s(1 - s)$ with $s \in \mathbb{C}$. Then there is a unique such eigenfunction $\omega_s(z, z_0)$ that extends continuously to z_0 with $\omega(z_0, z_0) = 1$, it is y_0^{-s} times the radial symmetrization of y^s about z_0 and it is defined globally.

Proof. 1. Let f be such an eigenfunction. By (3.22), $f = h \circ r$ on V for some smooth $h : \mathbb{R}_{>0} \to \mathbb{R}$. Note that r is smooth on a punctured neighborhood of z_0 by (D.37). By (3.25), solving $\Delta f + \lambda f = 0$ is equivalent to solving $Dh = 0$ for some differential operator D of degree 2, with nonzero highest degree coefficient at $t > 0$ sufficiently small. So by (F.18), local solutions extend globally and we can calculate the dimension.

2. Note that $\mathbf{H}$ is isotropic, so the result from 1. applies. By (3.29), y^s is an eigenfunction with eigenvalue $s(1-s)$. Let us check that its radial symmetrization is still an eigenfunction. Indeed, by (F.32) Δ is G-invariant so that by (3.19), taking the symmetrization commutes with Δ. Finally, if we divide by y_0^s we obtain a normalized $\omega_s(\cdot, z_0)$, with $\omega_s(z_0, z_0) = 1$.

   It remains to show that this is the unique eigenfunction that extends continuously to z_0. This is done by explicitly writing Δ in polar coordinates, as in done in [Bump, 1996, Proposition 2.3.4]. □

Applying the correspondence from (3.21) to $\omega_s(z, z_0)$, we obtain a point-pair invariant ω_s of eigenvalue $s(1-s)$ for $-\Delta$. We may thus reformulate the conclusion as follows: there is, up to scalar, a unique point-pair invariant on $\mathbf{H}$ with eigenvalue $s(1-s)$. It has the property that $\omega_s(z, z) = 1$ for all z. For $z_1 \in \mathbf{H}$ arbitrary, $\omega_s(\cdot, z_1)$ satisfies the properties from (3.30)(2) with z_0 replaced by z_1. This implies that ω_s does not depend on the choice of z_0.

Proposition 3.31 (Selberg eigenfunction principle)**.** Let $s \in \mathbb{C}$ and $f \in C^\infty(\mathbf{H})$ be an eigenfunction of $-\Delta$ with eigenvalue $s(1-s)$. Then f is an eigenfunction for any compactly supported point-pair invariant k:

$$(k \star f)(z) = (k \star \omega_s(\cdot, z))(z) \cdot f(z)$$

Proof. Fix $z \in \mathbf{H}$, then we have $(k \star f)_z^{\mathrm{rad}} = k \star f_z^{\mathrm{rad}}$ by (3.26)(2). The symmetrization f_z^{rad} is still an eigenfunction with the same eigenvalue by (3.19).5, so $f_z^{\mathrm{rad}}(w) = f(z)\omega_s(w, z)$ by uniqueness of spherical eigenfunctions. That is,

$$(k \star f)_z^{\mathrm{rad}}(w) = (k \star f_z^{\mathrm{rad}})(w) = f(z) \cdot (k \star \omega_s(\cdot, z))(w)$$

Evaluating in $w = z$ gives the result. Since z was arbitrary, we are done. □

Accordingly, we define:

Definition 3.32. Let k be a point-pair invariant on $\mathbf{H}$ with compact support. Fix any $z_0 \in \mathbf{H}$ and define the *Selberg-transform*

$$\begin{aligned} \widehat{k}(s) &= (k \star \omega_s(\cdot, z_0))(z_0) \\ &= (k \circ \omega_s(\cdot, \cdot))(z_0, z_0) \\ &= (k \star y^s)(z_0) \cdot y_0^{-s} \end{aligned}$$

where $\circ$ is the convolution product from (3.14).

The Selberg eigenfunction principle then reads

$$k \star f = \widehat{k}(s) \cdot f$$

Proposition 3.33 (Properties of the Selberg transform)**.** 1. The Selberg-transform $k \mapsto \widehat{k}(s)$ is an algebra homomorphism $A(\mathbf{H}) \to \mathcal{O}(\mathbb{C})$ to the ring of entire functions.

2. For all k, $\widehat{k}(s) = \widehat{k}(1-s)$.

Proof. 1. Immediate. 2. Because ω_{1-s} has the same eigenvalue as ω_s. □

Proposition 3.34 (The Selberg transform of approximations of the identity)**.** Let ρ_δ be a point-pair invariant approximation of the identity (3.8) for small $\delta > 0$. Then:

1. $\widehat{\rho_\delta}(s) \to 1$ locally uniformly in s.

2. $\widehat{-\Delta\rho_\delta}(s) \to s(1-s)$ locally uniformly in s.

Proof. 1. Fix $w_0 \in \mathbf{H}$. We have

$$\widehat{\rho_\delta}(s) = y_0^{-s}(\rho_\delta \star y^s)(w_0)$$
$$= y_0^{-s} \int_{\mathbf{H}} \rho_\delta(w_0, w) y^s dw$$

The support of the integral is contained in some small geodesic ball $\overline{B}(y_0, \delta)$. Inside the integral, we approximate y^s by y_0^s:

$$|\widehat{\rho_\delta}(s) - 1| = \left| y_0^{-s} \int_{\overline{B}(y_0,\delta)} \rho_\delta(w_0, w)(y^s - y_0^s) dw \right|$$
$$\leqslant |y_0^{-s}| \int_{\overline{B}(y_0,\delta)} \rho_\delta(w_0, w)|y - y_0||s| \max(y^{\sigma-1}, y_0^{\sigma-1}) dw$$

where we used the mean value inequality for y^s. Let $I_\delta \subseteq \mathbb{R}$ be the interval

$$I_\delta = \Im\mathrm{m}\, \overline{B}(y_0, \delta) = [y_0 e^{-\delta}, y_0 e^{\delta}]$$

We see that for $w \in \overline{B}(y_0, \delta)$ the difference $|y - y_0|$ is small, in fact $O(\delta)$ for $\delta \to 0$. Thus as long as s stays in a compact set K, we have

$$|\widehat{\rho_\delta}(s) - 1| \ll \delta \cdot \max_{\substack{s \in K \\ y \in I_\delta}} \left| s y^{\sigma-1} y_0^{-s} \right|$$

and the convergence follows.

2. Recall that y^s is an eigenfunction of $-\Delta$ with eigenvalue $s(1-s)$ (3.29). Thus

$$\widehat{-\Delta\rho_\delta}(s) y^s = (-\Delta\rho_\delta \star y^s)$$
$$= -\Delta(\rho_\delta \star y^s)$$
$$= -\widehat{\rho_\delta}(s)\Delta(y^s)$$
$$= \widehat{\rho_\delta}(s) s(1-s) y^s$$

Thus $\widehat{-\Delta\rho_\delta}(s) = \widehat{\rho_\delta}(s)s(1-s) \to s(1-s)$ locally uniformly. □

4 Functions on the quotient $\Gamma\backslash\mathbf{H}$

Armed with many tools to deal with smooth functions on symmetric spaces such as $\mathbf{H}$, we begin our study of Real analytic Eisenstein series. Fix a lattice $\Gamma \subset G$. Unless otherwise stated, we will assume that it has only one cusp, at ∞, and that its stabilizer is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We will often use the notation $\sigma = \Re\mathrm{e}(s)$.

4.1 Eisenstein series

Note how, for $s \in \mathbb{C}$, y^s is invariant under Γ_∞. We define the real analytic Eisenstein series by summing the images of y^s right cosets of Γ_∞:

Definition 4.1 (Real analytic Eisenstein series)**.**

$$E(w,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma w)^s$$

The series is Γ-invariant at those s for which it converges absolutely. We will drop 'real analytic' and simply refer to it as the 'Eisenstein series'.

Proposition 4.2. The series $E(w,s)$ converges uniformly and absolutely on compact subsets of $\mathbf{H} \times \{\sigma > 1\}$.

We first give a proof for $\Gamma = \mathsf{PSL}_2(\mathbb{Z})$, based on ideas from [Charollois, 2017, Proposition 1.1]. Then by Bézout's theorem we have a bijection

$$\begin{aligned} \Gamma_\infty \backslash \Gamma &\longrightarrow \{(c,d) \in \mathbb{Z}^2 : \gcd(c,d) = 1\}/\{\pm 1\} \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto (c,d) \end{aligned} \tag{4.3}$$

Proof. Let $K \subseteq \mathbf{H}$ be compact. For each $w \in K$, $|cw+d|^2$ is a positive definite quadratic form in $(c,d) \in \mathbb{R}^2$, hence positive on the unit ball for the sup norm on $\mathbb{R}^2$. That is, $|cw+d| \geqslant C \cdot \max(|c|,|d|)$ with C independent of $(c,d) \in \mathbb{R}^2 - \{0\}$. By continuity and compactness of K, we may assume $C = C(K)$ independent of w.
Now for $R \in \mathbb{N}$ there are $\ll R$ pairs $(c,d) \in \mathbb{Z}^2 - \{0\}$ with $\max(|c|,|d|) = R$. We deduce that

$$2|E(w,s)| \leqslant \sum_{(m,n) \in \mathbb{Z}^2 - \{0\}} \frac{y^\sigma}{|cw+d|^{2\sigma}} \ll y^\sigma \sum_{R=1}^{\infty} \frac{R}{R^{2\sigma}}$$

and the conclusion follows. □

The proof relied crucially on the bijection (4.3). In general, we don't have such a nice description of $\Gamma_\infty\backslash\Gamma$, and we have to exploit the discretenesss of Γ in a different way. The proof below uses the arguments from [Iwaniec, 2002, Lemma 2.10], where one can also find explicit bounds.

Proof in the general case. Let $K \subseteq \mathbf{H}$ be compact. There exists $\delta > 0$ such that K lies above the horizontal line $\Im\mathrm{m}\, w = \delta$. For the trivial coset $\gamma_0 \in \Gamma_\infty\backslash\Gamma$, we have $\Im\mathrm{m}(\gamma_0 w) = \Im\mathrm{m}(w)$, for all other cosets we have that $\Im\mathrm{m}(\gamma w) \leqslant 1/(\delta c_\infty^2)$ is bounded, by (2.22). We want them to be small. Let $w = x + iy$. We have $\gamma(w) = y|cw+d|^{-2}$, so we want $|cw+d|^2$ to be large, as before. Let $M \geqslant 1$ and consider the set

$$S(M) = \{\gamma \in \Gamma_\infty\backslash\Gamma - \{\gamma_0\} : |cw+d|^2 \leqslant M\}$$

We want to bound its cardinality. For $\gamma \in S(M)$, the bound on the imaginary part of $cw+d$ implies $c \leqslant M^{1/2}y^{-1} \leqslant M^{1/2}\delta^{-1}$, and the bound on the real part implies $|cx+d|^2 \leqslant M$. Now observe that, when $\gamma, \gamma' \in S(M)$ are distinct, then

$$\gamma(\gamma')^{-1} = \begin{pmatrix} * & * \\ cd' - dc' & * \end{pmatrix} \in \Gamma_\infty\backslash\Gamma - \{\gamma_0\}$$

so that $|cd' - dc'| \geqslant c_\infty$, that is,

$$\left|\frac{d}{c} - \frac{d'}{c'}\right| \geqslant \frac{c_\infty}{cc'} \quad , \quad \left|x + \frac{d}{c}\right| \leqslant \frac{M^{1/2}}{c} \qquad \forall \gamma \neq \gamma'$$

That is, the elements of $S(M)$ correspond to fractions d/c that lie in a bounded interval around $x \in \mathbb{R}$ and such that each two of them are at least a certain distance apart. Estimating naively, using $c_\infty \leqslant c \ll M^{1/2}$, gives that $\#S(M) \ll 1 + M^{1/2}(M^{1/2})^2 \ll M^{3/2}$. Here we bounded the size of the gaps from below by M^{-1}. But we can do better: take a dyadic partition of the interval $[c_\infty, M^{1/2}\delta^{-1}]$, say $[2^n c_\infty, 2^{n+1}c_\infty]$ for $0 \leqslant n \leqslant \frac{1}{2}\log_2 M + O(1)$. Then in such an interval, fractions d/c are at distance at least $c_\infty^{-1}(2^{n+1}c_\infty)^2$ and so it contains at most

$$\ll 1 + \frac{M^{1/2}}{c_\infty 2^n} \cdot \frac{(2^{n+1}c_\infty)^2}{c_\infty} \ll M^{1/2}2^n$$

such fractions. Summing over n, we obtain

$$\#S(M) \ll M^{1/2} \cdot M^{1/2} = M$$

We saved ourselves a factor $M^{1/2}$. We conclude by

$$\begin{aligned} |E(w,s)|y^{-\sigma} &\leqslant 1 + \sum_{\gamma \in \Gamma_\infty \backslash \Gamma - \{\gamma_0\}} \frac{1}{|cw+d|^{2\sigma}} \\ &\leqslant 1 + \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} \#S(n+1) \\ &\ll \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma-1}} \end{aligned}$$

Note that with the estimate $S(M) \ll M^{3/2}$ we can only show convergence for $\sigma > \frac{5}{4}$. □

We give another geometric proof, which consists of estimating each term of the Eisenstein series by an integral over a small domain, and piecing those domains together. The argument is makes it more intuitive why converges for $\sigma > 1$.

Second proof in the general case. [Cohen and Sarnak, 1980, Corollary 1.7]For $\delta > 0$, let k_δ be a point-pair invariant supported on points at distance $\leqslant \delta$ and with values in $[0,1]$. We could take an approximation of the identity, but all we care about is that it has small support and that its Selberg-transform $\widehat{k}_\delta(s)$ is nonzero at real arguments. By the Selberg eigenfunction principle,[3]

$$\widehat{k}_\delta(\sigma)y_2^\sigma = \int_{\mathbf{H}} k_\delta(w,z)y_1^\sigma \frac{dx_1dy_1}{y_1^2} \leqslant \int_{d(z,w)<\delta} y_1^\sigma \frac{dx_1dy_1}{y_1^2} \tag{4.4}$$

where we denote

$$z = x_1 + iy_1 \quad , \quad w = x_2 + iy_2$$

Let $w_0 \in \mathbf{H}$. Because Γ acts discontinuously, its orbit is discrete. Let $\delta_0 > 0$ be such that the closed geodesic ball $\overline{B}(w_0,\delta_0)$ is disjoint from its translates $\gamma\overline{B}(w_0,\delta_0) = \overline{B}(\gamma w_0,\delta_0)$ for $\gamma \notin \Gamma_{w_0}$. Then $\overline{B}(\gamma w_0,\delta_0)$ is disjoint from $\overline{B}(\mu w_0,\delta_0)$ for $\gamma\mu^{-1} \notin \Gamma_{w_0}$. Then for all $w \in \overline{B}(w_0,\delta_0/2)$, we have the same relation with δ_0 replaced by $\delta = \delta_0/2$:

$$\overline{B}(\gamma w,\delta) \cap \overline{B}(\mu w,\delta) = \varnothing \quad , \quad \gamma\mu^{-1} \notin \Gamma_{w_0}$$

Fix the standard fundamental domain F for Γ, contained in the standard fundamental domain $F_\infty = [0,1] \times \mathbb{R}_{>0}$ for Γ_∞. Let S be a set of representatives for $\Gamma_\infty\backslash\Gamma$. Suppose first that Γ_{w_0} is trivial. Then

[3]Since we're only interested in an inequality, we don't actually need the Selberg eigenfunction here: we could directly estimate $y_2^\sigma \ll \int_{d(z,w)<\delta} y_1^{\sigma-2}dx_1dy_1$ with an implicit constant depending on y_2 and σ in a controlled way. That way, one can avoid the use of point-pair invariants.

when γ runs through s, the images of the smaller balls $B(\gamma w, \delta)$ in the quotient $\Gamma_\infty\backslash\mathbf{H}$ are disjoint. Let $\epsilon > 0$ such that at least one larger ball $\overline{B}(\gamma w_0, \delta_0)$ lies just above the line $\Im\mathrm{m}\, w = \epsilon$. By the estimate (2.9), it lies below the line $\Im\mathrm{m}\, w = \epsilon e^{2\delta_0}$. And by (2.22), all other larger balls lie below the line $\Im\mathrm{m}\, w = (c_\infty^2\epsilon)^{-1}$. Let $M = \max(\epsilon e^{2\delta_0}, (c_\infty^2\epsilon)^{-1})$. Because the images of the larger balls in the quotient $\Gamma_\infty\backslash\mathbf{H}$ are disjoint, we have, for $w \in B(w_0, \delta)$:

$$\begin{aligned}\sum_{\gamma\in S}(\gamma w)^\sigma &\leqslant \sum_{\gamma\in S}\sup_{w\in B(w_0,\delta)}\Im\mathrm{m}(\gamma w)^\sigma\\ &\leqslant \frac{1}{\widehat{k}_\delta(\sigma)}\sum_{\gamma\in S}\int_{z\in B(\gamma w_0,\delta_0)} y_1^\sigma\frac{dx_1dy_1}{y_1^2}\\ &\leqslant \frac{1}{\widehat{k}_\delta(\sigma)}\int_{F_\infty\cap\{\Im\mathrm{m}\, z\leqslant M\}} y_1^\sigma\frac{dx_1dy_1}{y_1^2}\end{aligned}$$

which converges for $\sigma > 1$. The absolute and uniform convergence follows now from Weierstrass's M-test.

Suppose now that Γ_{w_0} is not trivial, i.e. that w_0 is an elliptic fixed point. Then the stabilizer is still finite, say of size m. Then S can be partitioned in at most m subsets S_i such that for all i and $\gamma, \mu \in S_i$ we have $\gamma\mu^{-1} \notin \Gamma_{w_0}$. The same argument as before now shows that

$$\begin{aligned}\sum_{\gamma\in S}\sup_{w\in B(w_0,\delta)}\Im\mathrm{m}(\gamma w)^\sigma &\leqslant \sum_{i=1}^m\sum_{\gamma\in S_i}\sup_{w\in B(w_0,\delta)}\Im\mathrm{m}(\gamma w)^\sigma\\ &\leqslant m\cdot\frac{1}{\widehat{k}_\delta(\sigma)}\int_{F_\infty\cap\{\Im\mathrm{m}\, z\leqslant M\}} y_1^\sigma\frac{dx_1dy_1}{y_1^2}\end{aligned}$$

for some $M > 0$ depending on w_0 and δ_0. We conclude again using the M-test. □

Proposition 4.5 (Analytic properties of the Eisenstein series)**.**

1. For fixed $w \in \mathbf{H}$, $E(w, s)$ is holomorphic in s.
2. When we forget the complex structure of the second argument, $E(w, s)$ is jointly smooth (as a function on an open set of $\mathbb{R}^4$). In particular, for fixed s with $\sigma > 1$ it is smooth in w.
3. For fixed s with $\sigma > 1$, $E(w, s)$ is an eigenfunction of the Laplacian $-\Delta$ with eigenvalue $s(1-s)$.
4. For fixed s with $\sigma > 1$, $E(w, s)$ is real analytic in w.
5. $E(w, s)$ is jointly real analytic.

Proof.

1. By locally uniform convergence.
2. We have to show that the series of partial derivatives converges locally uniformly, for all higher order derivatives. We can factor out y^s. By the Cauchy–Riemann equations, it suffices to consider $\frac{d^k}{ds}\frac{\partial^l}{\partial y}\frac{\partial^m}{\partial x}(E(w,s)y^{-s})$, and by Hurwitz's theorem we can suppose $k = 0$. By induction,

   $$\frac{\partial^l}{\partial y}\frac{\partial^m}{\partial x}\left(|cw+d|^{-2s}\right) = s^{(l+m)}|cw+d|^{-2(s+l+m)}P_{l,m}(cx+d, cy, c)$$

   where $s^{(l+m)}$ denotes the rising factorial and $P_{l,m}$ is some polynomial of total degree at most $2(l+m)$. As long as y is bounded away from 0, we can bound

   $$\begin{aligned}P_{l,m}(cx+d, cy, c) &\ll (|cx+d| + |cy| + |c|)^{2(l+m)}\\ &\ll (|cx+d| + |cy|)^{2(l+m)}\\ &\ll |cw+d|^{2(l+m)}\end{aligned}$$

   reducing the convergence to that of the Eisenstein series, where any of the above proofs can be used.

3. Because we can now differentiate term-wise. The function y^s is an eigenfunction with that eigenvalue, and hence so is each $\Im\mathrm{m}(\gamma w)^s$ because $-\Delta$ is invariant under isometries.

4. By elliptic regularity (F.25).

5. By (5.44), itself a consequence of elliptic regularity for systems of equations. □

We give another proof of the smoothness and the fact that $E(w,s)$ is a Laplacian eigenfunction, which illustrates the power of convolution operators:

Proof. Let k be a compactly supported point-pair invariant on $\mathbf{H}$. By the Selberg eigenfunction principle, we have

$$k \star (\Im\mathrm{m}\,\gamma w)^s = \widehat{k}(s)\,\Im\mathrm{m}(\gamma w)^s$$

Because the Eisenstein series $E(w,s)$ is continuous in w and converges locally uniformly, dominated convergence implies

$$k \star E(\cdot,s) = \widehat{k}(s) E(\cdot,s) \tag{4.6}$$

Now, fix s. If k is smooth, then so is the LHS. If $\widehat{k}(s) \neq 0$, we conclude that $E(w,s)$ is smooth in w. To find such k, let k be an approximation of the identity, then by (3.34) $\widehat{k}(s)$ can be as close to 1 as we want. In particular, there exists k with $\widehat{k}(s) \neq 0$.

Now, note that (4.6) also implies that $E(w,s)$ is jointly smooth, because the variables w and s are separated in the LHS.

Now let $k = -\Delta\rho_\delta$ be the Laplacian applied to an approximation of the identity. We have[4] (4.6) on the one hand, and, by (3.27) for the action of the invariant differential operator $-\Delta$,

$$k \star E(\cdot,s) = \rho_\delta \star (-\Delta E(\cdot,s)) \tag{4.7}$$

Now let $\delta \to 0$; then (4.6) converges to $s(1-s)E(\cdot,s)$ by (3.34), and (4.7) converges locally uniformly to $-\Delta E(\cdot,s)$ by (3.9). □

Lemma 4.8. For $\sigma > 1$, we have $E(w,s) = y^s + O_s(y^2)$ as $y \to \infty$.

Proof. We look at the last proof of (4.2), and use the same notation. For $y_0 = \Im\mathrm{m}\,w_0$ sufficiently large, the stabilizer Γ_{w_0} is trivial, because there are only finitely many elliptic orbits under Γ. We are thus in the first case in that proof. By (2.22), for y_0 sufficiently large the ball $B(w_0,\delta_0)$ is disjoint from the balls $B(\gamma w_0,\delta_0)$ with $\gamma \notin \Gamma_\infty$ (it suffices to have $2\log(c_\infty y) > 2\delta_0$). For the argument to work, we also need it to be disjoint from those balls with $\gamma \in \Gamma_\infty$. Therefore it suffices that the real parts of points in $B(w_0,\delta_0)$ take values in an interval of length less than 1. We know that the imaginary part in this ball is at most $y_0 e^{\delta_0}$, so by (2.10) real parts differ by at most

$$(\exp(2\delta_0) - 1) \cdot 2(y_0 e^{\delta_0})^2$$

which is less than 1 for δ_0 of size $\asymp y_0^{-1}$. For large y_0, and $\gamma \notin \Gamma_\infty$, the balls $B(\gamma w_0,\delta_0)$ lie below the line $\Im\mathrm{m}\,w = 1$, so we have, for appropriate $\delta_0 \asymp y_0^{-1}$:

$$\begin{aligned} |E(w_0,s) - y_0^s| &\leqslant \sum_{\gamma \in S - \Gamma_\infty} \Im\mathrm{m}(\gamma w_0)^\sigma \\ &\leqslant \frac{1}{\widehat{k}_{\delta_0}(\sigma)} \sum_{\gamma \in S-\Gamma_\infty} \int_{z \in B(\gamma w_0,\delta_0)} y_1^\sigma \frac{dx_1 dy_1}{y_1^2} \\ &\leqslant \frac{1}{\widehat{k}_{\delta_0}(\sigma)} \int_{F_\infty \cap \{\Im\mathrm{m}\,z \leqslant 1\}} y_1^\sigma \frac{dx_1 dy_1}{y_1^2} \end{aligned}$$

[4] Again by dominated convergence, not by the Selberg eigenfunction principle, since we don't know yet that $E(\cdot,s)$ is a Laplacian eigenfunction!

The second factor is bounded independently of w_0. We have to bound the first. Let k_δ be the non-normalized approximation to the identity

$$k_\delta(z,w) = c_\delta^{-1}\rho_\delta = \begin{cases} \exp\left(-\dfrac{1}{1-\left(\frac{d(z,w)}{\delta}\right)^2}\right) & : d(z,w) < \delta \\ 0 & : d(z,w) \geqslant \delta \end{cases} \tag{4.9}$$

(compare with (3.8)). By (3.34), we have $\widehat{k}_\delta \sim c_\delta^{-1}$ as $\delta \to 0$, where

$$c_\delta^{-1} = \int_{B(z,\delta)} \exp\left(-\frac{1}{1-\left(\frac{d(z,w)}{\delta}\right)^2}\right) d\mu(w)$$

for all $z \in \mathbf{H}$. Fix z. We want to bound this from below. The integrand is $\gg 1$ on $B(z,\delta/2)$. Note that a ball $B(z,\epsilon)$ contains a Euclidean rectangle I_ϵ with sides of length $1/\epsilon$ by virtue of the formula (2.6) for $d(z,w)$ and the approximation

$$\operatorname{arcosh}(1+x) = \log\left(1+x+\sqrt{(x+1)^2-1}\right) = \log(1+O(\sqrt{x})) = O(\sqrt{x}) \qquad (x \geqslant 0)$$

We conclude that $c_\delta^{-1} \gg \operatorname{vol}(I_{\delta/2}) \asymp \delta^2$. Substituting this in our main estimate, we obtain

$$|E(w_0,s) - y_0^s| \ll_s \widehat{k}_{\delta_0}(\sigma)^{-1} \sim_s c_{\delta_0} \ll \delta_0^{-2} \asymp y_0^2$$ □

We are ready to apply the study of Fourier expansions from Section 2.4:

Theorem 4.10. The Eisenstein series has the Fourier expansion

$$E(w,s) = y^s + \phi(s)y^{1-s} + \sum_{l\in\mathbb{Z}-\{0\}} a_n(s) W_{0,s-1/2}(4\pi|n|y)e(nx) \tag{4.11}$$

valid for $w \in \mathbf{H}$ and $\sigma > 1$, for a certain holomorphic function ϕ and certain functions a_n.

Proof. First fix s. Then by the preceding lemma we have that (2.29) applies, which gives the shape of the nonconstant terms. By (2.28), the constant term has the form $b(s)y^s + \phi(s)y^{1-s}$. By the preceding lemma, $b(s) = 1$ for $\sigma > 2$. It remains to argue that ϕ, b and a_n are holomorphic, by uniqueness of analytic continuation it then follows that $b(s) = 1$ also for $1 < \sigma \leqslant 2$. The constant term equals

$$C(y,s) = \int_0^1 E(x+iy,s)dx$$

which is indeed holomorphic in s for fixed y. But we want to obtain holomorphy of each of the two terms $b(s)y^s$ and $\phi(s)y^{1-s}$. Note that b and ϕ do not depend on y, so taking y, y' for which the vectors (y^s, y^{1-s}) and (y'^s, y'^{1-s}) are linearly independent, we can solve for $b(s)$, $\phi(s)$ and conclude that they are holomorphic. □

The general asymptotics for Laplacian eigenfunctions of polynomial growth apply, and (2.30) gives:

Corollary 4.12. We have, for $w \in \mathbf{H}$:

$$E(w,s) = y^s + \phi(s)y^{1-s} + O_s(e^{-2\pi y}) \quad , \quad (y \to \infty)$$

Remark 4.13. While (4.8) only proves that the Eisenstein series is $y^s + O_s(y^2)$, the study of Fourier expansions and Witthaker functions gives us for free that it is, in fact, $y^s + O_s(y^{1-\sigma})$!

4.2 Automorphic kernels

Let Γ be a lattice which we assume $\mathsf{SL}_2(\mathbb{Z})$, unless otherwise stated. We want to study functions on the quotient $Y = \Gamma\backslash\mathbf{H}$, and thus, naturally, integral operators on this manifold. While a point-pair invariant on $\mathbf{H}$ sends Γ-invariant functions to Γ-invariant functions, it is itself not an integral kernel on $Y \times Y$. We thus define:

Definition 4.14. Let Γ be a (possibly cocompact) lattice in G. Let k be a continuous point-pair invariant of compact support on $\mathbf{H}$. Its automorphization is

$$K(z, w) = \sum_{\gamma\in\Gamma} k(z, \gamma w) \tag{4.15}$$

This is well defined by requiring that k has compact support: because Γ acts properly discontinuously, the sum is in fact finite on compact subsets of $\mathbf{H}\times\mathbf{H}$. Thus K is at least as smooth as k, is Γ-invariant in both variables and symmetric.
More precisely, for fixed z, the number of terms in the definition of $K(z, w)$ is at most the number of closed fundamental domains that intersect the compact set $\operatorname{supp} k(z, \cdot)$. Or more generally, for a compact set L, the support of the restriction $k|_{L\times\mathbf{H}}$ is compact, and the number of terms is bounded independently of $z \in L$. In particular, when Y is compact the number of terms in the definition of K is bounded independently of z and w.
From now on we assume Γ is not cocompact. One can wonder whether it is still the case that the number of terms in the definition of K is bounded. The answer is no, but we do have a good upper bound. Throughout this subsection we will denote

$$z = x_1 + iy_1 \quad , \quad w = x_2 + iy_2$$

Proposition 4.16. For $y_1 \to \infty$, we have

$$K(z, w) \ll_{\Gamma,k} y_1 \tag{4.17}$$

uniformly in w.

That is, we gain automorphy of the kernel at the cost of having to work with an unbounded (possibly not even square-integrable) kernel.

Proof. Because k has compact support, it is bounded, so it suffices to estimate the number of terms in the definition of $K(z, w)$.
Say k is supported on points at distance at most R. By the lower bound (2.9) for $d(z, w)$, we have that the support $\operatorname{supp} k(z, \cdot)$ is contained in the horizontal strip $\mathbb{R} \times [y_1 e^{-R}, y_1 e^{R}]$. In particular, for y_1 sufficiently large, only horizontal translates of the standard fundamental domain F can intersect the support $\operatorname{supp} k(z, \cdot)$. But how many? We have from (2.6) that $d(z, w) \leqslant R$ implies $(x_2 - x_1)^2 \leqslant 2y_1y_2(\cosh(R) - 1)$, thus there are at most $\ll y_1$ such fundamental domains. □

We will recover this estimate using the Fourier expansion, in (4.31).

Remark 4.18 (Large imaginary parts)**.** Suppose k is supported on point pairs (z, w) at distance at most R. As we have seen in the proof above, this implies that there exists a constant $c > 0$ such that $k(z, w) = 0$ unless $y_1/c < y_2 < y_1 c$. We show that the same holds for K as a function on $Y \times Y$: let $z, w \in F$ lie in the standard fundamental domain. In the sum $K(z, w) = \sum_{\gamma\in\Gamma} k(z, \gamma w)$, only the therms with $\Im\mathrm{m}(\gamma w) > y_1/c$ give a nonzero contribution.
Now note that by (2.22), for every $\epsilon > 0$ there exists $A > 0$ depending on Γ such that if $\Im\mathrm{m}\, v > A$, then either $\gamma \in \Gamma_\infty$ or $\Im\mathrm{m}(\gamma v) < \epsilon$. Take ϵ sufficiently small so that F lies above the line $\Im\mathrm{m}\, v = \epsilon$. Then for $y_1/c > A$ and $w \in F$, we cannot have $\Im\mathrm{m}(\gamma w) > y_1/c$ unless $\gamma \in \Gamma_\infty$ and w already has imaginary part $y_2 > y_1/c$.
The kernel K is symmetric, so by changing the roles of z and w, we similarly have $y_1 > y_2/c$ if $(z, w) \in \operatorname{supp} K$ with $y_2/c > A$. We conclude that:

$$y_1 \asymp_k y_2 \quad , \quad (z, w) \in F \cap \operatorname{supp} K$$

In particular, for a fixed K and $z, w \in F$, the expression

$$\text{“}y \to \infty\text{”}$$

is unambiguous when it is understood that $(z, w) \in \operatorname{supp} K$. It means that both $y_1, y_2 \to \infty$, or equivalently, at least one of them. We will use this abuse of notation throughout this section. Finally, we note that for $y_1 > Ac$ or $y_2 > Ac$ we have

$$\begin{aligned} K(z,w) &= \sum_{\gamma\in\Gamma} k(z,\gamma w) \\ &= \sum_{\gamma\in\Gamma_\infty} k(z,\gamma w) \end{aligned} \tag{4.19}$$

Because K is Γ-invariant in both variables, we can consider it as a kernel on $Y \times Y$, and we have:

Proposition 4.20. For $f : \mathbf{H} \to \mathbb{C}$ measurable and Γ-invariant, which descends as $\bar{f} : \Gamma\backslash\mathbf{H} \to C$, we have:

$$K \star_Y \bar{f} = \overline{k \star f} \tag{4.21}$$

Proof. By unfolding: let F be any fundamental domain for Γ. Fix $z \in \mathbf{H}$, then

$$\begin{aligned} \int_Y K(z,w)\bar{f}(w)d\mu(w) &= \int_F K(z,w)\bar{f}(w)d\mu(w) \\ &= \int_F \sum_{\gamma\in\Gamma} k(z,\gamma w)f(w)d\mu(w) \\ &= \sum_{\gamma\in\Gamma}\int_F k(z,\gamma w)f(\gamma w)d\mu(\gamma w) \\ &= \sum_{\gamma\in\Gamma}\int_F k(z,w)f(w)d\mu(w) \\ &= \int_{\mathbf{H}} k(z,w)f(w)d\mu(w) \end{aligned}$$

by noting that the sum over $\gamma \in \Gamma$ is secretly a finite sum. □

For non-cocompact lattices Γ, the convolution operator K is not necessarily Hilbert–Schmidt. But using the estimate (4.17), we find:

Proposition 4.22. The automorphization K defines a bounded self-adjoint convolution operator on $L^2(Y)$.

Proof. Let F be the standard fundamental domain, so that $y_1 \asymp y_2$ uniformly for $z, w \in F$ and $(z, w) \in \operatorname{supp} K$. First, for fixed $z \in Y$ we have:

$$\begin{aligned} (K \star f)(z) &= \int_F K(z,w)f(w)\frac{dx_2dy_2}{y_2^2} \\ &\ll \int_{\{y_1\asymp y_2\}} \frac{|f(w)|}{y_2}dx_2dy_2 \end{aligned} \tag{4.23}$$

where we restrict the domain of integration, because letting the last integral run over F may not give a finite value. We are tempted to say that, by Cauchy-Schwarz

$$|(K \star f)(z)|^2 \ll \int_{\{y_1\asymp y_2\}} dx_2dy_2 \int_{\{y_1\asymp y_2\}} \frac{|f(w)|^2}{y_2^2}dx_2dy_2$$

where the second factor is bounded by $\|f\|_2^2$, and the first factor is as small as $\ll y_1$. But we are not happy, because integrating $y_1 \|f\|_2^2$ against the hyperbolic measure $\frac{dx_1 dy_1}{y_1^2}$ does not give a finite value. The problem is that estimating the second integral by $\|f\|_2^2$ is too crude. Estimating directly $\|K \star f\|_2$ instead, we have, switching the order of integration:

$$\|K \star f\|_2^2 \ll \int_F y_1 \int_{\{y_1 \asymp y_2\}} \frac{|f(w)|}{y_2^2} dx_2 dy_2 \frac{dx_1 dy_1}{y_1^2}$$
$$= \int_F \frac{|f(w)|}{y_2^2} \int_{\{y_1 \asymp y_2\}} \frac{dx_1 dy_1}{y_1} dx_2 dy_2$$

The inner integral is $O(1)$, and the boundedness follows. □

If we want to make the bound more explicit, we see from the proof and that of the upper bound (4.17) that when k is supported on point pairs at distance $\leqslant R$, then

$$\|K\| \ll \|k\|_\infty \cdot g(R) \tag{4.24}$$

for some fixed continuous function $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ of exponential growth. In particular, when the support of k does not increase, the upper bound depends only on $\|k\|_\infty$.

4.2.1 Fourier expansions

Consider a smooth compactly supported point-pair invariant k and its automorphization $K(z, w)$ on $\Gamma\backslash\mathbf{H}$. For fixed z, it is in particular Γ_∞-invariant in w. Fourier-expanding the function $K(z, w + x)$ for fixed (z, w) and evaluating it in $x = 0$ gives:

Proposition 4.25. We have

$$K(z, w) = \sum_{n \in \mathbb{Z}} \widehat{K}_n(z, w) \tag{4.26}$$

where

$$\widehat{K}_n(z, w) = \int_0^1 K(z, w + x) e(-nx) dx$$

We see that the $\widehat{K}_n$ are smooth functions in (z, w). They are Γ-invariant in z and Γ_∞-invariant in w. We call $\widehat{K}_0$ the *constant term*.
As for the Fourier expansion of smooth functions $\mathbb{R} \to \mathbb{C}$, we know that the Fourier coefficients $\widehat{K}_n$ are rapidly decreasing as $|n| \to \infty$: We have the general bound

$$|\widehat{K}_n(w, z)| \leqslant |2\pi n|^{-p} \int_0^1 \left| \frac{\partial^p K(z, w + x)}{\partial x^p} \right| dx \qquad \forall p > 0, n \neq 0$$

Note how the Poincaré metric $\frac{dx^2 + dy^2}{y^2}$ tends to 0 as $y \to \infty$. Thus the function k, which depends only on the distance, gets "spread out" more and more as its arguments approach $i\infty$, and we expect its derivatives to go to 0:

Proposition 4.27 (Approximating an automorphic kernel by its constant term)**.** For $y \to \infty$ we have

$$K(z, w) = \widehat{K}_0(z, w) + O_N(y^{-N}) \tag{4.28}$$

Proof. We start from the Fourier expansion (4.26). From (4.19) we have

$$K(z, w) = \sum_{\gamma \in \Gamma_\infty} k(z, \gamma w)$$

for y large enough. Because the sum is now restricted to $\gamma \in \Gamma_\infty$ we can unfold and obtain:

$$\begin{aligned}\widehat{K}_n(z,w) &= \int_0^1 \sum_{\gamma\in\Gamma_\infty} k(z,\gamma(w+x))e^{-2\pi inx}dx \\ &= \int_{\mathbb{R}} k(z,w+x)e^{-2\pi inx}dx\end{aligned}$$

By (3.22) we have that k is a smooth function of the hyperbolic distance. Say $k(z,w) = \Phi(u(z,w))$, where

$$u(z,w) = \frac{|z-w|^2}{y_1y_2}$$

as in (2.7) and Φ is smooth on $\mathbb{R}$. Then

$$D_x k(z,w+x) = \Phi'(u(z,w+x))\frac{\partial(|z-w-x|^2)}{\partial x}\cdot\frac{1}{y_1y_2}$$

where the second factor is a degree 1 polynomial in x. We see a factor $(y_1y_2)^{-1}$ appear, which we think of as small. When we differentiate again, we don't just get one term with a factor $(y_1y_2)^{-2}$, there is also a term with only a factor $(y_1y_2)^{-1}$ which comes from differentiating the polynomial. Keeping track of all terms, we obtain by induction that $D_x^N k(z,w+x)$ is a (finite) sum of functions of the form

$$\Phi^{(b)}(u(z,w+x))\cdot P(z,w,x)\frac{1}{(y_1y_2)^b}$$

with $b \leqslant M$, P a polynomial of certain degree a in x and at most $2a$ in x_1,x_2,y_1,y_2.
For every such term we have $2b-a = N$: indeed, when differentiating it, we either differentiate $\Phi^{(b)}$, which increases b by 1 and increases a by 1, or we differentiate the polynomial, which keeps b constant and decreases a by 1. In either case, we see that $2b-a$ increases by 1. In particular from $2b-a=N$ we have $2(b-a) = N+a \geqslant N$. Since we may assume x_1,x_2 bounded (w.l.o.g. z and w lie in the standard fundamental domain) each such term is bounded by

$$\begin{aligned}\ll_N |\Phi^{(b)}(u(z,w+x))|\frac{(y_1^2+y_2^2)^a}{(y_1y_2)^b} &\ll |\Phi^{(b)}(u(z,w+x))|\cdot y^{2a-2b} \\ &\leqslant |\Phi^{(b)}(u(z,w+x))|\cdot y^{-N}\end{aligned}$$

where we denote y for either y_1 or y_2, this abuse of notation being justified by (4.18).
For the Fourier coefficients, we now have for all $N>0$:

$$\begin{aligned}|\widehat{K}_n(z,w)| &\leqslant \frac{1}{(2\pi|n|)^N}\int_{\mathbb{R}}|D_x^N k(z,w+x)|dx \\ &\ll_N \frac{1}{|ny|^N}\int_{\mathbb{R}}|\Psi(u(z,w+x))|dx\end{aligned}$$

where $\Psi = \sum_{b\leqslant N}|\Phi^{(b)}|$ has compact support. It remains to bound the integral. By the lemma below, it is $\ll_N y$. We conclude that

$$K(z,w) - \widehat{K}_0(z,w) \ll_N \frac{1}{y^{N-1}}\sum_{n\neq 0}\frac{1}{|n|^N} \ll \frac{1}{y^{N-1}} \qquad \square$$

Lemma 4.29. Let $\Phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be a continuous compactly supported function, and $z,w \in \mathbf{H}$. Then, with u as in (2.7):

$$\int_{\mathbb{R}}\Phi(u(z,w+x))dx = (y_1y_2)^{1/2}\int_{\mathbb{R}}\Phi\left(x^2+\frac{y_2}{y_1}+\frac{y_1}{y_2}-2\right)dx \tag{4.30}$$

where the integrand is bounded by an integrable function independently of z and w.

Proof. We have

$$u(z, w + x) = \frac{|z - w - x|^2}{y_1 y_2} = \frac{(x_1 - x_2 - x)^2 + (y_1 - y_2)^2}{y_1 y_2}$$

so that substituting $t = (x + x_2 - x_1)/(y_1 y_2)^{1/2}$ gives (4.30). Finally, say Φ is supported on $[0, A]$, and say $|\Phi| \leqslant B$, then we have the upper bound

$$\int_{\mathbb{R}} \Phi \left(x^2 + \frac{y_2}{y_1} + \frac{y_1}{y_2} - 2 \right) dx \leqslant 2 \int_0^{\sqrt{A}} B dx < \infty \qquad \square$$

In particular, we recover the estimate (4.17) by applying this to the constant term $\widehat{K}_0$: as $y \to \infty$ we have, with $k = \Phi \circ u$:

$$\begin{aligned} \widehat{K}_0(z, w) &= \int_{\mathbb{R}} k(z, w + x) dx \\ &= (y_1 y_2)^{1/2} \int_{\mathbb{R}} \Phi \left(x^2 + \frac{y_2}{y_1} + \frac{y_1}{y_2} - 2 \right) dx \end{aligned} \tag{4.31}$$

where the integral is bounded by a constant.
From (4.20), automorphized kernels behave well with respect to convolution. We can refine this as follows:

Proposition 4.32. Let k be a compactly supported point-pair invariant with automorphization K, and $f : \mathbf{H} \to \mathbb{C}$ be smooth and Γ-invariant, with constant term C_f. Then

$$(\widehat{K}_0 \star_F f)(z) = (k \star C_f)(z) \quad , \quad y_1 > B \tag{4.33}$$

where the first convolution is on the standard fundamental domain F, the second is on $\mathbf{H}$, and B is a constant depending on Γ and k.

Recall that the Fourier coefficients $\widehat{K}_n$ are not Γ-invariant in the second variable, so it does not make sense to view $\widehat{K}_n$ as a convolution operator on the quotient $\Gamma \backslash \mathbf{H}$. Note also how we regard $C_f(z)$ here as a function of z, while it is (by definition) constant on horizontal lines.

Proof. Let $B > 0$ be large enough so that (4.19) holds for $y_1 > B$. We have, by unfolding,

$$\begin{aligned} (\widehat{K}_n \star_F f)(z) &= \int_F \widehat{K}_n(z, w) f(w) d\mu(w) \\ &= \int_F \int_0^1 K(z, w + x) f(w) dx d\mu(w) \\ &= \int_F \int_0^1 \sum_{\gamma \in \Gamma_\infty} k(z, \gamma(w + x)) f(w) dx d\mu(w) \\ &= \int_F \int_0^1 \sum_{\gamma \in \Gamma_\infty} k(z, \gamma w + x) f(w) dx d\mu(w) \\ &= \int_F \int_0^1 \sum_{\gamma \in \Gamma} k(z, \gamma w + x) f(w) dx d\mu(w) \\ &= \int_{\mathbf{H}} \int_0^1 k(z, w + x) f(w) dx d\mu(w) \\ &= \int_{\mathbf{H}} k(z, w) \int_0^1 f(w + x) dx d\mu(w) \end{aligned} \qquad \square$$

One might wonder why we don't prove that

$$(\widehat{K}_n \star_F f)(z) = (k \star \widehat{f}_n)(z)$$

for all $n \in \mathbb{Z}$ and y_1 sufficiently large. The only thing that prevents us is a subtlety in the definition of $\widehat{K}_n$: we Fourier-expanded $K(z, w+x)$ for fixed (z, w) and obtain a Fourier series (4.26) without the oscillating factor $e(nx)$. While for f, we Fourier expanded $f(x+iy)$ for fixed y, and not $f(x+iy+x')$ for fixed (x, y), so we do get the factor $e(nx)$. In the end, it is of little importance which way we define the Fourier coefficients, and we will care very little about $\widehat{K}_n$ for $n \neq 0$.

4.2.2 Truncated kernels

We have seen that automorphic kernels increase sufficiently slowly at the cusp so that they define a bounded convolution operator. In various situations it is desirable to have a compact operator acting on $L^2(Y)$. From (4.28) we see that it is the constant term $\widehat{K}_0$ that prevents K from being square-integrable. We want to subtract that constant term. Our criteria are:

1. We want a kernel that is automorphic in both variables.
2. It has to be smooth.

Inspired by $K - \widehat{K}_0$, we construct a smooth automorphic kernel that looks like it. Now, $\widehat{K}_0$ is only Γ_∞-invariant in the second variable, so that $\widehat{K}_0 f$ need not be Γ-invariant when f is. In general, there are three options to make a non-automorphic f automorphic:

1. The 'method of images': Take the sum of its images under the action of Γ, as we did when defining the Eisenstein series:
$$\sum_{\gamma \in \Gamma} f(\gamma z)$$
This preserves smoothness, but has the disadvantage that convergence can be painful to show.

2. Restrict f to the standard fundamental domain F, and translate it to other fundamental domains:
$$[f]_F(z) := \sum_{\gamma \in \Gamma} f|_F(\gamma^{-1} z) \quad , \quad z \in \gamma F$$
This is well-defined except on a measure 0 set (the boundary ∂F and its translates). We can make it smooth and everywhere defined by letting a compactly supported point-pair invariant act on it; the result will still be Γ-invariant (4.20).

3. A variation of the second method: if we assume that $f(z)$ is invariant under Γ_∞, then so is $\alpha(y) f(z)$ where we take $\alpha \in C^\infty(\mathbb{R})$ of the form
$$\alpha(y) = \begin{cases} 0 & : y \leqslant A \\ 1 & : y \geqslant A+1 \end{cases}$$
Now $\alpha(y) f(z)$ is supported on points with large imaginary part. Taking A sufficiently large so that there are no elliptic fixed points with imaginary part $\geqslant A$, then
$$[\alpha(y) f]_F$$
is everywhere defined and smooth: there is no need to let a point-pair invariant act on it. Indeed: it is smooth in a neighborhood of F, because the transformation that maps F to an adjacent fundamental domain F', fixes (setwise) the intersection $F \cap F'$, so that $[\alpha(y) f]_F$ is zero in a neighborhood of $F \cap F'$.

We thus define, following [Iwaniec, 2002, §4.2] resp. [Brumley, 2015, §5.8] resp. [Cohen and Sarnak, 1980, p. 23]:

Definition 4.34. For a compactly supported point-pair invariant k and smooth Γ-invariant f, define

$$\begin{aligned} L_1 f &= K \star_Y f - \left(\sum_{\gamma\in\Gamma} \int_0^1 k(\cdot, \gamma\cdot + t)dt \right) \star_Y f \\ L_2 f &= k \star [k \star f - k \star C_f]_F \\ L_3 f &= [k \star f - \alpha(y_1) \cdot k \star C_f]_F \end{aligned} \tag{4.35}$$

None of these operators has a reason to be self-adjoint. It would have been more natural to define

$$L_1' f = K \star_Y f - \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \widehat{K}_0(\cdot, \gamma\cdot) \star_Y f$$

but that sum need not converge, essentially because it gives a double sum over $\gamma \in \Gamma_\infty\backslash\Gamma$, another one coming from the definition of K. It would also have been more natural to define

$$L_2' f = k \star [(K - \widehat{K}_0) \star_F f]_F$$

which is almost the same, by (4.33), and we will make the comparison precise. We stay with the former definition of L_2 simply to respect the definition from the source.
We check that each of these defines a compact convolution operator on $L^2(Y)$.

1. For L_1 we have to check that the sum converges locally uniformly. Indeed, k has compact support, and the argument is exactly the same as for the convergence and smoothness of (4.15). For y_1 large, we have

$$\begin{aligned} H_1(z,w) &:= \sum_{\gamma\in\Gamma} \int_0^1 k(z, \gamma w + t)dt \\ &= \sum_{\gamma\in\Gamma_\infty} \int_0^1 k(z, \gamma w + t)dt \\ &= \sum_{\gamma\in\Gamma_\infty} \int_0^1 k(z, \gamma(w + t))dt \\ &= \sum_{\gamma\in\Gamma} \int_0^1 k(z, \gamma(w + t))dt = \widehat{K}_0(z,w) \end{aligned}$$

   so that L_1 is a compact convolution operator by the estimate (4.28).

2. For L_2, we note that by (4.33), for $z \in F$ with $y_1 > C = C(k, \Gamma)$ sufficiently large,

$$(k \star f - k \star C_f)(z) = ((K - \widehat{K}_0) \star_F f)(z)$$

   where the second convolution is on F. Thus $f \mapsto k \star f - k \star C_f$ is the convolution operator on F defined by the kernel

$$\begin{aligned} &\chi_{\{y_1 > C\}}(K(z,w) - \widehat{K}_0(z,w)) \\ &+ \chi_{\{y_1 \leqslant C\}} \left(K(z,w) + \widehat{K}_0(z,w) - \int_0^1 \sum_{\gamma\in\Gamma} k(z, \gamma w + t)dt \right) \\ &=: K(z,w) - H_2(z,w) \end{aligned}$$

   We know from the bound (4.28) that the term supported on $y_1 > C$ is bounded. The term supported on $y_1 \leqslant C$ is compactly supported on $F \times F$. This bounded kernel makes $K - H_2$ a compact convolution operator on $L^2(F)$. Hence so is $L_2 = K \circ (K - H_2)$.

3. For L_3, we find, for $z \in F$:

$$(k \star f - \alpha(y_1) \cdot k \star C_f)(z) = K \star_Y f - \alpha(y_1) \sum_{\gamma \in \Gamma} \int_0^1 k(z, \gamma w + t) f(w) dt$$
$$=: K \star_Y f - H_3 \star_Y f$$

   That is, we see that H_1 and H_3 coincide for large values of y_1, hence their kernels differ by a compactly supported, bounded function on $Y \times Y$. We conclude that L^3 is compact.

   Note that the kernel H_3 need not be smooth in the second variable on Y. We can fix this by choosing a larger value of A in the definition of α, depending on the support of k.

Each of the integral operators L_i has a kernel that is rapidly decreasing. Thus not only do they define compact operators, they also send functions of polynomial growth to L^2 functions.
Since we have only modified our kernels a little bit, we expect the Selberg eigenfunction principle to hold approximately:

Proposition 4.36. Let f be a Laplacian eigenfunction on $\Gamma\backslash\mathbf{H}$ with eigenvalue $s(1-s)$. Then

$$\begin{aligned} (L_j - \widehat{k}(s))f &= -H_j \star f \quad , \quad j \in \{1,3\} \\ (L_j - \widehat{k}(s)^2)f &= -K \star H_j \star f \quad , \quad j = 2 \end{aligned} \tag{4.37}$$

where for $z \in F$ with y_1 sufficiently large,

$$\begin{aligned} H_j \star f &= K \star C_f \quad , \quad j \in \{1,3\} \\ K \star H_j \star f &= K \star K \star C_f \quad , \quad j = 2 \end{aligned} \tag{4.38}$$

Proof. From the compatibility relation (4.20) we have that $K \star f = \widehat{k}(s) f$, and the first identity follows. For the second, we note that $k \circ k$ has Selberg-transform $\widehat{k}(s)^2$. □

4.3 Maass forms

Let $\Gamma \subset G$ be any lattice, and denote $Y = \Gamma\backslash\mathbf{H}$ for the quotient, as before. We do not assume that Γ has only one cusp, in order to illustrate some subtle points.

Definition 4.39. When ∞ is a cusp for Γ and f is Γ-invariant, we say f

1. is of *polynomial growth* at ∞ if there exist $N > 0$ such that

$$f(x+iy) \ll y^N \qquad (y \to \infty)$$

   uniformly in x.

2. *vanishes* at ∞ if

$$\lim_{y \to \infty} f(x+iy) = 0$$

   uniformly in x.

When $\mathfrak{a}$ is any cusp for Γ, and $\sigma_\mathfrak{a} \in G$ is such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$, then ∞ is a cusp for $\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}$. When f is Γ-invariant, we say f

1. is of *polynomial growth* at $\mathfrak{a}$ if the $\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{a}$-invariant function $\sigma_\mathfrak{a}^{-1} \circ f \circ \sigma_\mathfrak{a}$ is of polynomial growth at ∞.

2. *vanishes* at $\mathfrak{a}$ if $\sigma_\mathfrak{a}^{-1} \circ f \circ \sigma_\mathfrak{a}$ vanishes at ∞.

Definition 4.40. A *Maass form* (of weight 0) for Γ is a real analytic $f \in C^\infty(\mathbf{H}, \mathbb{C})$ that is:

1. an eigenfunction for $-\Delta$.

2. Γ-invariant

3. of polynomial growth at all cusps.

We denote by $H(\Gamma, \lambda)$ the space of Maass forms with eigenvalue λ, and by $H^c(\Gamma, \lambda)$ the subspace of Maass cusp forms. [5]

Note that, by elliptic regularity, if f is a C^2 eigenfunction for $-\Delta$, it is automatically real analytic (F.25). If Γ is cocompact, there are no cusps and $H(\Gamma, \lambda) = H^c(\Gamma, \lambda)$ is a full eigenspace of Δ.

Example 4.41. The Eisenstein series $E(w, s)$ lives in $H(\Gamma, s(1-s))$ for all $\sigma > 1$.

4.3.1 Cusp forms

Recall that Maass forms with eigenvalue $s(1-s)$ admit a Fourier expansion w.r.t. every cusp, whose constant term C_f is a linear combination of y^s and y^{1-s} (2.28).

Proposition 4.42. Let $\lambda \in \mathbb{C}$ and $f \in H(\Gamma, \lambda)$, consider the following statements:

(a) $C_f^{\mathfrak{a}} = 0$ in the Fourier expansion at every cusp $\mathfrak{a}$.

(b) $f \in H^c$

(c) $f \in L^2$.

We always have (a) $\implies$ (b) $\implies$ (c). For $\lambda \neq 0$ the three are equivalent, and for $\lambda = 0$ we have

$$H^c(\Gamma, \lambda) = \{f : C_f^{\mathfrak{a}} = 0, \forall \mathfrak{a}\} \subsetneq H(\Gamma, \lambda) \cap L^2(Y)$$

where the inclusion is strict when there are cusps.

Proof. We try to work at every cusp separately: is f is square-integrable in a neighborhood of a cusp $\mathfrak{a}$ iff it vanishes at $\mathfrak{a}$? W.l.o.g. suppose $\mathfrak{a} = \infty$. Let $\lambda = s(1-s)$ with $\sigma \geqslant \frac{1}{2}$. We know f has a Fourier expansion of the form

$$f(x+iy) = C_f(y) + R(x+iy)$$

where R is exponentially decreasing at ∞, uniformly in x, and $C_f = c_1 g_1 + c_2 g_2$ is a linear combination of

$$\begin{cases} g_1(y) = y^s \text{ and } g_2(y) = y^{1-s} & : s \neq \frac{1}{2} \\ g_1(y) = y^s \text{ and } g_2(y) = y^s \log y & : s = \frac{1}{2} \end{cases}$$

In particular, R is L^2 in a neighborhood of ∞ and vanishes at ∞.

(a) $\implies$ (b),(c): Immediate, from the Fourier expansion $f = C_f + R$.

(b) $\implies$ (c): R vanishes at ∞, so if f vanishes at ∞ then so does C_f, so that $c_1 = 0$ and also $c_2 = 0$ if $\sigma \leqslant 1$. In particular, the constant term contains only terms of the form y^ζ or $y^\zeta \log y$ with $\zeta < \frac{3}{2}$. This implies that C_f is square-integrable at ∞ against the hyperbolic measure $\frac{dxdy}{y^2}$.

(c) $\implies$ (a): Suppose f is L^2 in a neighborhood of ∞. Unless we know something about s, we cannot simply conclude from the Fourier-expansion that f vanishes at ∞: if for example $\frac{1}{2} < \sigma \leqslant 1$, there could be a term y^{1-s} which is L^2 but does not vanish at ∞. We want to invoke spectral theory to conclude that $\lambda \in \mathbb{R}$. We could study the restriction of the Laplacian to smooth functions on a neighborhood of ∞, but nothing guarantees that it has only nonnegative eigenvalues: the Laplacian is positive on *complete* manifolds (G.22).

Hence if we suppose that f is *globally* L^2, we know that λ is real. Then $\sigma = \frac{1}{2}$ or $s = 1$. If $s = \frac{1}{2}$, we deduce that $c_1 = c_2 = 0$, because g_1 and g_2 have different growth and do not vanish at ∞.

[5] This is probably not the standard notation, if there exists one. Idt is inspired by a notation for the space of Harmonic Maass forms (hence the 'H').

If $s = 1$, we have $c_1 = 0$, and we see that f is a constant function plus the function R which vanishes at ∞. If ∞ is the only cusp, we can conclude that

$$H(\Gamma, \lambda) \cap L^2(Y) = H^c(\Gamma, \lambda) \oplus \mathbb{C}$$

If there are multiple cusps, the situation is more mysterious. We observe that constant functions are in L^2 without being cusp forms, but there might be other exceptions. We leave the case $\lambda = 0$ at that.

If $s \neq \frac{1}{2}$ with $\sigma = \frac{1}{2}$, there is a subtlety: we have to verify that the oscillations of y^s and y^{1-s} do not resonate in such a way that a nontrivial linear combination of them can be L^2. This could only possibly happen if $|c_1| = |c_2|$: otherwise, one term dominates the other. Let $-c_1/c_2 = \exp(i\theta)$ and $s = \frac{1}{2} + it$. Using the bound $\exp(iT) - 1 \gg d(T, 2\pi\mathbb{Z})$ we have

$$\begin{aligned} \left|c_2^{-1} C_f(y)\right| &= y^{1/2} \left|e^{i(\theta + t\log y)} - 1\right| \\ &\gg y^{1/2} \chi \left\{ y : \theta + t \log y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] + 2\pi\mathbb{Z} \right\} \end{aligned}$$

so that $\|C_f\|_{L^2}^2$ is at least the measure of the set appearing in the characteristic function. But this set contains infinitely many pairwise disjoint intervals of length 1 (say), hence $C_f \notin L^2$. Contradiction. □

Note how we only needed 'local' information near a cusp to conclude that if f vanishes at a cusp, then it is L^2 near the cusp. For the other implication, we needed global information: the proof does not exclude Maass forms that are L^2 at some cusp and are of nontrivial growth at that cusp.

In the proof we saw that, by spectral theory of the Laplacian, there can only be cusp forms if $\lambda \in \mathbb{R}_{\geqslant 0}$:

$$H^c(\Gamma, \lambda) = 0 \quad , \quad \lambda \notin \mathbb{R}_{\geqslant 0}$$

This implies that for $\lambda \notin \mathbb{R}_{\geqslant 0}$ a Maass form is determined by its constant term, and even by the asymptotics of its constant term:

Proposition 4.43. If Γ has cusps, $f \in H(\Gamma, \lambda)$ and $g \in H(\Gamma, \mu)$ with $f \sim g \gg 1$ as $y \to \infty$ at every cusp, then $\lambda = \mu$. If $\lambda = \mu \notin \mathbb{R}_{\geqslant 0}$, then $f = g$.

Proof. Say ∞ is a cusp, and let $\lambda = s(1-s)$ and $\mu = \zeta(1-\zeta)$ with $\sigma = \Re\mathrm{e}\, s, \Re\mathrm{e}\, \zeta \geqslant \frac{1}{2}$. By assumption, f is not a cusp form so that by the Fourier expansion there exists $c, d \in \mathbb{C}$ such that $f \sim c \cdot y^s$ for $\sigma > \frac{1}{2}$, $f \sim cy^{1/2} \log y$ or $f \sim cy^{1/2}$ for $s = \frac{1}{2}$ and $f \sim (cy^{it} + dy^{-it})y^{1/2}$ if $s \neq \sigma = \frac{1}{2}$. Thus λ is determined by the asymptotics at ∞, and the first statement follows.

Moreover, we see that the constant terms of f and g at every cusp are equal up to possibly a term y^{1-s} for $\sigma > \frac{1}{2}$, $y^{1/2}$ for $s = \frac{1}{2}$ and 0 if $s \neq \sigma = \frac{1}{2}$. In any case, f and g are equal up to an L^2 Maass form. Hence if $\lambda \notin \mathbb{R}_{\geqslant 0}$, it follows that $f - g = 0$. □

In particular we obtain:

Proposition 4.44 (Uniqueness principle)**.** Suppose Γ has one cusp, ∞. Then for $\sigma > 1$, the Eisenstein series $E(w, s)$ is the unique function f such that

1. $f \in H(\Gamma, s(1-s))$
2. $f \sim y^s$ as $y \to \infty$.

Equivalently, it is the unique Maass form with

1. $(\Delta + s(1-s))f = 0$
2. $C_f = y^s + \phi(s)y^{1-s}$ for some $\phi(s) \in \mathbb{C}$.

4.3.2 Dimensions

We discuss the dimension of $H(\Gamma,\lambda)$ and $H^c(\Gamma,\lambda)$. As a start, we can refine the result from (4.42) by bounding the codimensions of the various spaces.

Proposition 4.45. If N is the number of cusps of Γ, then:

1. For $\lambda = s(1-s)$ with $\sigma > 1$:
$$\dim_{\mathbb{C}} H(\Gamma,\lambda) \leqslant N$$
2. For $\lambda \in \mathbb{R}_{>0}$ (i.e. $\sigma = \frac{1}{2}$):
$$\dim_{\mathbb{C}} \frac{H(\Gamma,\lambda)}{H(\Gamma,\lambda)\cap L^2(Y)} = \dim_{\mathbb{C}} \frac{H(\Gamma,\lambda)}{H^c(\Gamma,\lambda)} \leqslant 2N$$
3. For $\lambda = 0$ (i.e. $s = 1$):
$$\dim_{\mathbb{C}} \frac{H(\Gamma,\lambda)}{H(\Gamma,\lambda)\cap L^2(Y)} \leqslant \dim_{\mathbb{C}} \frac{H(\Gamma,\lambda)}{H^c(\Gamma,\lambda)} \leqslant \dim_{\mathbb{C}} \frac{H(\Gamma,\lambda)}{H(\Gamma,\lambda)\cap L^2(Y)} + N \leqslant 2N$$

Proof. As in the proof of (4.42), we have from the Fourier expansion that a Maass form is determined modulo L^2 Maass forms by the coefficients of the two functions g_1, g_2 appearing in its constant term. For $\sigma > 1$, at each cusp there is only one such coefficient that vanishes precisely on (the nonexistent) L^2 Maass forms: the coefficient of y^s. That is, we have an injective linear map

$$H(\Gamma,\lambda) = \frac{H(\Gamma,\lambda)}{H(\Gamma,\lambda)\cap L^2(Y)} \hookrightarrow \mathbb{C}^N$$

by sending a Maass form to the coefficients of y^s in its N Fourier expansions. The first inequality follows.
For $\lambda \in \mathbb{R}_{>0}$, the coefficients of both g_1 and g_2 vanish precisely on cusp forms, and the second statement follows similarly. For $\lambda = 0$, there are the coefficients of y^1 that vanish precisely on L^2 forms, which gives the rightmost inequality. An L^2 harmonic Maass form is determined modulo cusp forms by the coefficients of y^0 in its constant terms, hence

$$\dim_{\mathbb{C}} \frac{H(\Gamma,\lambda)\cap L^2(Y)}{H^c(\Gamma,\lambda)} \leqslant N$$

and the middle inequality follows. The leftmost inequality follows from the inclusion $H^c \subseteq H \cap L^2$. □

Theorem 4.46. All the spaces $H(\Gamma,\lambda)$ are finite-dimensional.

We give various proofs. Note that by (4.45) it is equivalent to show that H^c or $H \cap L^2$ is finite dimensional.

Proof 1, compact case. When Γ is cocompact, we know by a general result on Riemannian manifolds, that the eigenspaces of the Laplacian are finite-dimensional (G.28). That is, all $H(\Gamma,\lambda)$ have finite dimension. □

Proof 2, general case. When Γ is noncompact, one can show, by analyzing the resolvent of the hyperbolic Laplacian, that the L^2-eigenvalues of $-\Delta$ still go to infinity, counting multiplicities (G.29). In particular,
$$L^2(Y) \cap H(\Gamma,\lambda)$$
has finite dimension. The conclusion follows. □

We give a proof that does not use the fact that the Laplacian has compact resolvent.

Proof 3, compact case. Assume again that there are no cusps. Fix $\lambda = s(1-s)$. Let k be a point-pair invariant on $\mathbf{H}$ and K its automorphization (4.15). By the Selberg eigenfunction principle and (4.20), each $f \in H(\Gamma, \lambda)$ is an eigenfunction for K with eigenvalue $\widehat{k}(s)$. Because Y is compact, K is automatically a Hilbert–Schmidt integral operator on $L^2(Y)$, and thus compact. Its eigenspaces for nonzero eigenvalues are finite-dimensional, either by the general spectral theory of compact operators (A.35) or by the spectral theorem for compact self-adjoint operators (A.49). In particular, if $\widehat{k}(s) \neq 0$ then

$$\dim_{\mathbb{C}} H(\Gamma, \lambda) < \infty$$

Hence given s it suffices to find k with $\widehat{k}(s) \neq 0$. This is possible: we can let k be an approximation of the identity (3.34). □

By modifying the kernel K, we can make the argument work for noncompact quotients as well:

Proof 4, general case. Take k and K as before. When Y is noncompact, K has no reason to be a compact operator. Suppose first that there is only one cusp, which we may assume is ∞. We carefully select one of the compact truncated kernels from (4.35). Suppose a truncation L coincides with K on a linear subspace V of $L^2(Y)$. Because L^2-eigenspaces of L corresponding to nonzero eigenvalues μ are finite-dimensional, it follows that the μ-eigenspace of K restricted to V is finite-dimensional. Take $L = L_3$, so that L coincides with K on the space of Maass forms f with $C_f = 0$. From the proof of (4.42), we see that this includes in particular the $H^c(\Gamma, \lambda)$. Fix such a $\lambda = s(1-s) \in \mathbb{R}$ and take k an approximation of the identity, so that $\mu = \widehat{k}(s) \neq 0$. The μ-eigenspace of K restricted to $V = \{f : C_f = 0\}$ has finite-dimension; in particular $H^c(\Gamma, \lambda)$ has finite dimension. We might as well have taken $L = L_2$, which coincides with the compact operator $K \circ K$ on cusp forms, and whose Selberg transform is $\widehat{k}(s)^2$.
In the case of multiple cusps, the proof is similar. The only complication is that one has to truncate the kernel K at all cusps. □

We give another proof, which does not use spectral theory, except for relying on the fact that cusp forms have constant term of their Fourier expansions equal to 0, for which we have used the positivity of the Laplacian. It relies on the Baire category theorem via a mysterious lemma below.

Proof 5, general case. [Borel, 1997, Theorem 8.5] Take $\lambda \geqslant 0$. We show that $H^c(\Gamma, \lambda)$ is closed in $L^2(Y)$, so that its finite-dimensionality follows from the general lemma below. Take a sequence (f_n) of cusp forms in this eigenspace, which converges in L^2 to f. We show that f is a cusp form and a Laplacian eigenfunction with the same eigenvalue. Because Y has finite volume, it converges in L^1, and thus in distribution: for compactly supported $\phi \in C_0^\infty(Y)$, by dominated convergence:

$$\int_Y f_n \phi \to \int_Y f \phi$$

This shows that f is, in the distributional sense, an eigenfunction of $-\Delta$ with eigenvalue λ:

$$\int_Y f \cdot (\Delta + \lambda)\phi = 0 \quad , \quad \phi \in C_0^\infty(Y)$$

By elliptic regularity for weak solutions (F.26), it follows that f is smooth. Consequently, f is a Laplacian eigenfunction in the strong sense.
It remains to show that f is a cusp form. Because $\lambda \in \mathbb{R}_{\geqslant 0}$, this is equivalent to f having no constant term in the Fourier expansion w.r.t. every cusp. The argument is from [Borel, 1997, Proposition 8.2]. Fix a cusp, w.l.o.g. ∞, with stabilizer generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let C_f denote the constant term in the Fourier expansion of f. Suppose $C_f(y_0) \neq 0$ for some $y_0 > 0$. take $A > 0$ such that $[0,1] \times [A, \infty]$ is contained in the standard fundamental domain, so that the projection $\mathbf{H} \to Y$ is injective on it. We may suppose $y_0 > A$. Take a compactly supported $\phi \in C^\infty(\mathbb{R}_{>0})$ for which $\int_A^\infty C_f \phi \neq 0$. That is,

$$\int_0^1 \int_A^\infty f(x+iy)\phi(y) dy dx \neq 0$$

At the same time, by assumption:

$$\int_0^1 \int_A^\infty f_n(x+iy)\phi(y)dydx = 0 \qquad \forall n$$

By the choice of A, this integral expression is L^1-continuous in f: we have

$$\left| \int_0^1 \int_A^\infty g(x+iy)\phi(y)dydx \right| \leqslant \|g\|_1 \|\phi\|_\infty \quad , \quad g \in L^1(Y)$$

But $f_n \to f$ in L^1, a contradiction. □

Lemma 4.47. Let Z be a locally compact Hausdorff space with a positive finite measure μ. Let V be a closed subspace of $L^2(Z)$ contained in $L^\infty(Z)$. Then V is finite-dimensional.

The proof is so elegant that we cannot omit it. Note: we are not just requiring that V is closed in $L^2(Z) \cap L^\infty(Z)$.

Proof. [Borel, 1997, Lemma 8.3] Because Z has finite volume, $\|f\|_2 \leqslant \mu(Z) \|f\|_\infty$ for all measurable f, so we have a continuous inclusion $L^\infty(Z) \hookrightarrow L^2(Z)$, which restricts to a continuous bijection $j : (V, \|\cdot\|_\infty) \xrightarrow{\sim} (V, \|\cdot\|_2)$. Because V is closed in $L^2(Z)$, it is closed in $L^\infty(Z)$, and this is a continuous bijection between Banach spaces. By the open mapping theorem (A.13), the inverse of j is continuous.[6] Let $c > 0$ such that

$$\|f\|_\infty \leqslant c \cdot \|f\|_2 \quad , \quad (f \in V)$$

Suppose $v_1, \ldots, v_n$ are pairwise orthonormal functions in V. For all $a_1, \ldots, a_n \in \mathbb{C}$ we have

$$\left| \sum a_i v_i(z) \right| \leqslant c \cdot \left\| \sum a_i v_i \right\|_2 = c \cdot \left(\sum |a_i|^2 \right)^{1/2} \quad , \quad \text{(a.e. } z\text{)}$$

Taking $a_i = \overline{v_i(z)}$ we have, for all $z \in Z$:

$$\sum |v_i(z)|^2 \leqslant c^2 \quad , \quad \text{(a.e. } z\text{)}$$

Integrating over z gives

$$n \leqslant c^2 \mu(Z)$$

Hence $\dim V \leqslant c^2 \mu(Z)$. □

[6]This is the most mysterious part: it relies on Baire's theorem, whose proof for general complete metric spaces uses the axiom of choice.

5 Analytic continuation of Eisenstein series

Before embarking on various proofs of analytic (better: meromorphic) continuation of real analytic Eisenstein series in the sense of Theorem 1.1, we recall that there is a plenitude of notions of holomorphy and meromorphy: We will use the terminology from Appendix B rather freely. As in the previous section, we assume that Γ has one cusp, at ∞, whose stabilizer is generated by the parabolic element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

5.1 Elementary proofs

There are a few methods which apply (as far as is known) only to specific lattices. We do not present them in full detail here; the reader is invited to consult the references.

5.1.1 Proof by Poisson summation

Take $\Gamma = \mathsf{PSL}_2(\mathbb{Z})$. Recall the bijection

$$\begin{aligned} \Gamma_\infty \backslash \Gamma &\longrightarrow \{(c,d) \in \mathbb{Z}^2 : \gcd(c,d) = 1\}/\{\pm 1\} \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto (c,d) \end{aligned} \tag{5.1}$$

due to Bézout's theorem. Summing over all pairs $(c,d) \in \mathbb{Z}^2$ we obtain, for $\sigma = \Re\mathrm{e}(s) > 1$:

$$2\zeta(2s)E(w,s) = \sum_{(m,n)\in\mathbb{Z}^2} \left(\frac{|mw+n|^2}{y} \right)^{-s}$$

By grouping the pairs (m,n) by the value of $(m,n)/\gcd(m,n)$. This looks a lot like the Riemann zeta function:

$$2\zeta(2s) = \sum_{n\in\mathbb{Z}-\{0\}} (n^2)^{-s}$$

We know how to meromorphically continue the ζ-function, using Poisson summation for the *Jacobi theta function*: Define

$$\theta(t) := \sum_{n\in\mathbb{Z}} e^{-t\pi n^2} \qquad (t > 0)$$

Poisson summation gives the functional equation

$$\theta(1/t) = \sqrt{t} \cdot \theta(t)$$

One then expresses the ζ-function in terms of θ by

$$\pi^{-s}\gamma(s)\zeta(2s) = \int_0^\infty \left(\frac{\theta(t) - 1}{2} \right) t^s \frac{dt}{t}$$

The functional equation for θ then provides both the meromorphic continuation of ζ, the locations and order of its poles as well as the functional equation.
More generally, one can attach a Jacobi theta function to a quadratic form, such as

$$(m,n) \mapsto \frac{|mw+n|^2}{y}$$

Poisson summation gives a functional equation for this theta function, and similarly to the proof for ζ, we deduce the meromorphic continuation of $E(w,s)$. For details, see e.g. [Garrett, 2011].

5.1.2 Proof by Fourier expansion

Recall that Laplacian eigenfunctions of polynomial growth admit a Fourier expansion that is well-understood, and this applies in particular to the Eisenstein series (4.10). Take again $\Gamma = \mathsf{SL}_2(\mathbb{Z})$. Thanks to the fact that this group has very nice structure, we can compute all terms of the Fourier expansion of $E(w,s)$ explicitly. One finds that the constant term equals

$$C_{E(w,s)} = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. The nonconstant terms are explicitly expressible in terms of Bessel functions. Assuming the meromorphic continuation of ζ, together with the meromorphic continuation of Bessel functions one can conclude from here. Moreover, the functional equation for ζ together with the functional equation for Bessel functions, provides the functional equation for $E(w,s)$. For details, see e.g. [Charollois, 2017, Théorème 6.1.1], [Brumley, 2015, §4.2].

One may wonder why we are not satisfied with these proofs. The main reason is that they don't generalize well to arbitrary lattices Γ, and more generally Eisenstein series for other groups, such as $\mathsf{SL}_n(\mathbb{R})$: The problem with the proof by Poisson summation is that general Γ have no nice arithmetic structure. And even when it has, Poisson summation is not always possible. For example, one can give a proof using Poisson summation for a specific class of Eisenstein series for $\mathsf{SL}_n(\mathbb{R})$ with the lattice $\Gamma = \mathsf{PSL}_n(\mathbb{Z})$; *minimal-parabolic* Eisenstein series. This was observed by Langlands; the argument can also be found in [Garrett, 2012b].

The issue with the proof by Fourier expansion is that we have a priori no way to meromorphically continue the holomorphic function $\phi(s)$ that appears in the constant term (4.10). Instead, we will *deduce* the meromorphic continuation (and functional equation) of $\phi(s)$ from the meromorphic continuation of $E(w,s)$.

5.2 Proof via Fredholm-theory

The following proof of meromorphic continuation, due to Selberg, is largely based on the lecture notes [Cohen and Sarnak, 1980]. The idea is to use the Selberg eigenfunction principle to write the Eisenstein series, on the half-plane $\{\sigma > 1\}$ where it is defined, as the solution to a Fredholm equation. We then prove that this Fredholm equation has a unique solution for s in a larger domain containing the half-plane $\{\sigma > 1\}$, and that the solution depends analytically on s.

5.2.1 A truncated Eisenstein series.

Fredholm theory is about L^2 functions. But $E(w,s)$ is not an L^2-function, so we want to 'truncate it', by subtracting its constant term. Its constant term is $y^s - \phi(s)y^{1-s}$. We know very little about $\phi(s)$, so we prefer not to subtract it, and leave it untouched together with the nonconstant terms: they constitute the part of $E(w,s)$ that is nontrivial to analytically continue. Luckily, that mysterious part of the constant term is in L^2. The first part, y^s, is not. We want to subtract it while preserving automorphy, so we proceed similarly to how we constructed the third truncated kernel L_3: Let $A > 0$ and define for $\Re\mathrm{e}\, s > 1$ and $w \in F$ in the standard fundamental domain:

$$\widetilde{E}(w,s) = E(w,s) - \alpha(y)y^s \tag{5.2}$$

where $\alpha \in C^\infty(\mathbb{R})$ is such that

$$\alpha(y) = \begin{cases} 0 & : y \leqslant A \\ 1 & : y \geqslant A+1 \end{cases}$$

and A is chosen large enough so that there are no elliptic fixed points with imaginary part $\geqslant A$, that is, so that

$$[\widetilde{E}(w,s)]_F$$

is smooth and Γ-invariant. It is still analytic in s for fixed y (and indeed, still jointly differentiable). Finding a meromorphic continuation of E is equivalent to finding one of $\widetilde{E}$. Note also that $\widetilde{E}$ is in L^2 for all s, as follows from (4.12).

5.2.2 A Fredholm equation

Now let k be a compactly supported point-pair invariant on $\mathbf{H}$ and K its automorphization. By the Selberg eigenfunction principle and (4.20), E is an eigenfunction on $\Gamma\backslash\mathbf{H}$ of the integral operator defined by K, with eigenvalue $\widehat{k}(s)$. We expect this to be approximately true for $\widetilde{E}$. We have:

$$(K-\widehat{k}(s))\widetilde{E} = -(K-\widehat{k}(s))(\alpha(y)y^s) \tag{5.3}$$

Now K is a kernel supported on point pairs at distance at most (say) R, so that $(K\star f)(z)$ depends only on the values $f(w)$ for $d(z,w)<R$. Because $d(z,w)\geqslant|\log(y_1/y_2)|$ by (2.9), we have

$$(K\star(\alpha(y)y^s))(z) = (K\star y^s))(z) =: \widehat{k}(s)y^s(z)$$

for $y_1>(A+1)e^R$, that is, for y_1 large enough. We conclude that the RHS in (5.3) is compactly supported, with support bounded independently of s.
We are ready to apply Fredholm theory:

Theorem 5.4. $\widetilde{E}(w,s)$, and thus the Eisenstein series $E(w,s)$, has an analytic continuation to $\{\Re\mathrm{e}\, s > 1/2\} - [\frac{1}{2},1]$. More precisely, it is pointwise holomorphic and jointly smooth. Moreover, $\widetilde{E}(w,s)$ is square-integrable for all such s.

Proof. Let k be a compactly supported point-pair invariant on $\mathbf{H}$ and K its automorphization. We want to solve (5.3) for $\widetilde{E}$ by inverting the operator $K-\widehat{k}(s)$. First, fix s. Recall that K is a bounded self-adjoint operator on the Hilbert space $L^2(\Gamma\backslash\mathbf{H})$ (4.22). When $\widehat{k}(s)$ is not in the spectrum of K, we can solve the equation and obtain[7]

$$\widetilde{E}(w,s) = (K-\widehat{k}(s))^{-1}(K-\widehat{k}(s))(\alpha(y)y^s) \in L^2(\Gamma\backslash\mathbf{H})$$

A regularity theorem for Fredholm equations (C.3) implies that it is smooth in w.
For which s can we do this? Because K is self-adjoint, it has real spectrum. Let $k=-\Delta\rho_\delta$, the Laplacian applied to an approximation of the identity. By (3.34), $\widehat{k}(s)$ converges locally uniformly to $s(1-s)$ as $\delta\to 0$. Write $s=\sigma+it$ and note that the imaginary part

$$\Im\mathrm{m}(s(1-s)) = t(1-2\sigma)$$

is strictly negative as long as $\sigma>\frac{1}{2}$ and s is not real. So if we fix $s_0\in\{\sigma>\frac{1}{2}\}-\mathbb{R}$ and take δ sufficiently small, then $K-\widehat{k}(s)$ is invertible for all s in an open neighborhood U of s_0. We are ready to apply Fredholm theory: the RHS of (5.3) has compact support in w, independently of s. By (C.5), the solution $\widetilde{E}$ we obtain, is analytic in $s\in U$ and jointly smooth.
Now note that the construction of $\widetilde{E}(w,s)$ for each s depends on the choice of k. We have to argue that we obtain a unique solution, i.e. that the solutions obtained on neighborhoods U of each s_0 glue together. There are two ways to see this:

1. For $k=-\Delta\rho_\delta$, we have $\widehat{k}(s)\to s(1-s)$ locally uniformly as $\delta\to 0$. We can write $\{\sigma>\frac{1}{2}\}-\mathbb{R}$ as an increasing union of relatively compact open sets (U_n), the smallest of which intersects the half-plane $\{\sigma>1\}$. Letting $\delta\to 0$ as $n\to\infty$, we obtain solutions on each of the U_n, which glue together by uniqueness of analytic continuation for fixed w.

2. We can first construct an analytic continuation on neighborhoods of s_0, for $\Re\mathrm{e}\, s_0 = 1$. Those neighborhoods overlap with the half-plane $\{\sigma>1\}$, hence either by uniqueness of the solution to the Fredholm equation or by uniqueness of analytic continuation, that solution must coincide with the already defined $\widetilde{E}(w,s)$ for $\sigma>1$. We want to proceed in this fashion, extending our solution $\widetilde{E}(w,s)$ bit by bit until we cover all of $\{\sigma>\frac{1}{2}\}-\mathbb{R}$. But care must be taken when gluing those solutions, because uniqueness of analytic continuation does not apply to such successive

[7] Note that we cannot simply say that the operator $K-\widehat{k}(s)$ cancels with its inverse: the RHS of (5.3) must not be read as the bounded operator $K-\widehat{k}(s)$ applied to the function $\alpha(y)y^s$. Indeed, the latter is not in L^2!

extensions. (Unlike in the previous argument, where in each step the larger domain contains all the smaller ones.) Instead, we seek to apply uniqueness of the solution to the Fredholm equation.

Suppose we have analytically continued $\widetilde{E}$ to an open connected set V containing the half-plane $\{\sigma > 1\}$, and that there is a point s_0 on the boundary of V which is contained in the open set $\{\sigma > \frac{1}{2}\} - \mathbb{R}$. Fredholm theory gives us a point-pair invariant k, an open connected neighborhood U of s_0 and a solution $\overline{E}(w,s)$ of (5.3) on U. We want to show that it coincides with the already found $\widetilde{E}$ on $U \cap V$. Therefore, we need that $\widetilde{E}$ satisfies (5.3) for the point-pair invariant k (and not just some other point-pair invariant).

We know that $E(w,s)$ is jointly smooth, and the solution to our Fredholm equations, as well. Hence so is $\widetilde{E} + \alpha(y)y^s$. By uniqueness of analytic continuation and joint smoothness, it is still a Laplacian eigenfunction with eigenvalue $s(1-s)$. Thus the Selberg eigenfunction principle applies for $s \in U \cap V$. In particular, $\widetilde{E}$ satisfies (5.3) for the point-pair invariant k, on $U \cap V$. Uniqueness of the solution of the Fredholm equation implies that $\overline{E}$ coincides with $\widetilde{E}$.

Finally, we must argue that we can cover all of $\{\sigma > \frac{1}{2}\} - \mathbb{R}$ in this fashion. We can consider a maximal connected open set of $\{\sigma > \frac{1}{2}\} - \mathbb{R}$ on which $\widetilde{E}(w,s)$ is defined and apply Zorn's lemma. Or more elementary, take an arbitrary point s_0 in that domain, a compact segment linking it to the half-plane $\{\sigma > 1\}$, and use compactness of the segment to show that we can reach the point s_0 in a finite number of steps. □

Remark 5.5.

1. What is preventing us from analytically continuing beyond the line $\sigma = \frac{1}{2}$, is the spectrum of the operators K: they are not necessarily compact, hence might not have discrete spectrum. This makes that, for each s, we have to construct a point-pair invariant k whose Selberg-transform $\widehat{k}(s)$ avoids the real line (in a neighborhood of s). If the spectrum of K were discrete (with the exception of 0), it would suffice that $\widehat{k}(s)$ avoids 0, and we obtain a meromorphic continuation. Not surprisingly, this will be our strategy.

2. While the operators K are not compact, they are still bounded, and their spectrum is contained in the closed disk $\overline{B}(0, \|K\|)$. One can wonder whether, for s_0 fixed, we can find K such that $\widehat{k}(s)$ avoids just the spectrum of K in a neighborhood of s_0 (and not the entire real line). If, for example, the $k_\delta = -\Delta\rho_\delta$ from the proof give us K with $\|K\| \leqslant C$ bounded independently of δ, we would have an analytic continuation for all s such that $s(1-s)$ is not in the disk $\overline{B}(0, C)$. Unfortunately[8] this is not the case: the best bound we have for $\|K\|$ is (4.24): While the support of the k_δ is controlled, their sup norm is not.

5.2.3 Truncated kernels

We want to meromorphically continue beyond the line $\sigma = \frac{1}{2}$. As remarked above, we want a compact integral operator that sends $\widetilde{E}$ to an L^2 function whose analytic continuation is known, in order to apply Fredholm theory. Consider any of the truncated kernels L_i defined in (4.35). Let us fix one, say L_3. We want to view the equation (4.37), with $f = E(w,s)$, as a Fredholm equation:

$$\begin{aligned}(L_3 - \widehat{k}(s))E(w,s) &= -H_3 \star (y^s + \phi(s)y^{1-s}) \\ &= -\alpha(y)\widehat{k}(s)(y^s + \phi(s)y^{1-s})\end{aligned} \tag{5.6}$$

There are two issues:

1. The RHS is not in L^2 (as before).

2. The RHS involves the function ϕ, whose meromorphic continuation is not known.

We know how to deal with the first problem: we can modify the Eisenstein series and let $\widetilde{E}(w,s) = E(w,s) - \alpha(y)y^s$, just as before. The main problem is the function $\phi(s)$, which we know virtually nothing about. The strategy is as follows:

[8] And not surprisingly: otherwise the meromorphic continuation of the Eisenstein series would have only finitely many poles.

1. Solve (5.6) for all $s \in \mathbb{C}$, without the term involving ϕ in the RHS. Call the solution E^*.

2. Conclude, simply by taking linear combinations, that

$$E^*(w, s) + \phi(s)E^*(1 - s)$$

solves (5.6) with the full RHS, for $\sigma > 1$.

3. Conclude that $E^*(w, s) + \phi(s)E^*(1 - s)$ must coincide with $E(w, s)$ for $\sigma > 1$.

4. Exploit the fact that E is a Laplacian eigenfunction to derive a formula for ϕ in terms of E^*, which we use to meromorphically continue it.

The last step is the most mysterious one; it relies on spectral properties of the Laplacian. Note that, while $\widetilde{E}$ satisfies

$$(L_3 - \widehat{k}(s))\widetilde{E}(w, s) = -\alpha(y)\widehat{k}(s)(y^s + \phi(s)y^{1-s}) + (L_3 - \widehat{k}(s))(\alpha(y)y^s) \tag{5.7}$$

we never solve this equation directly. The problem is that it does not have enough symmetry: when we solve (5.7) without the term involving ϕ and call the solution $\widetilde{E}^*$, we cannot conclude that $\widetilde{E}^*(w, s) + \phi(s)\widetilde{E}^*(w, 1 - s)$ solves (5.7) with the full RHS. Instead, we define the more symmetric truncation $\overline{E}(w, s) = E(w, s) - \alpha(y)(y^s + \phi(s)y^{1-s})$ and solve

$$(L_3 - \widehat{k}(s))\overline{E}(w, s) = -\alpha(y)\widehat{k}(s)(y^s + \phi(s)y^{1-s}) + (L_3 - \widehat{k}(s))(\alpha(y)(y^s + \phi(s)y^{1-s})) \tag{5.8}$$

first without the terms involving ϕ, call the solution E^{**} and then conclude that

$$E^{**}(w, s) + \phi(s)E^{**}(w, 1 - s)$$

solves (5.8) with the full RHS.

Lemma 5.9 (Partial solution to the Fredholm equation)**.** For any compactly supported point-pair invariant for which $\widehat{k}(s)$ is not identically zero, the equation

$$(L_3 - \widehat{k}(s))E^{**}(w, s) = -\alpha(y)\widehat{k}(s)y^s + (L_3 - \widehat{k}(s))(\alpha(y)y^s)$$

has a L^2-meromorphic jointly smooth (away from poles) solution $E^*(w, s)$ for $s \in \mathbb{C}$.

Proof. The RHS is simply $L_3(\alpha(y)y^s)$, and equals $L_3 y^s = 0$ for large values of y. Thus the RHS is compactly supported with support bounded independently of s. We can apply Fredholm theory. Because L_3 is a convolution operator with bounded kernel, there are now two options:

1. Fredholm theory for integral operators with bounded kernel (C.5), (C.8).

2. The general Fredholm theorem for bounded operators (C.15).

The conclusion follows. □

Corollary 5.10. Let $E^{**}(w, s)$ be as in (5.9). Then for $\sigma > 1$,

$$\overline{E}(w, s) = E^{**}(w, s) + \phi(s)E^{**}(w, 1 - s)$$

That is, with $E^*(w, s) := E^{**}(w, s) + \alpha(y)y^s$ we have

$$E(w, s) = E^*(w, s) + \phi(s)E^*(w, 1 - s) \tag{5.11}$$

Proof. The first identity follows because both sides are L^2-solutions of (5.8), which has a unique solution for s in a suitable open set. The second follows by definition of E^* and $\overline{E}$. □

Because $E^{**}(w, s)$ is jointly smooth away from poles and L^2-meromorphic, it is C^∞-meromorphic (B.31), and hence so is $E^*(w, s)$. But it is worth remembering that $E^*(w, s)$ equals $\alpha(y)y^s$ plus an $L^2 \cap C^\infty$-valued meromorphic function.

5.2.4 Uniqueness principle

We recall that the Laplacian $-\Delta$ has a unique self-adjoint unbounded extension to $L^2(Y)$, whose domain contains the smooth L^2-functions, and which is still positive (G.22). This gave rise to the uniqueness principle (4.44): a smooth L^2 Laplacian eigenfunction must have a nonnegative real eigenvalue. The idea is to use this to extract $\phi(s)$ from (5.11).

Lemma 5.12. There exists a discrete set $S \subset \mathbb{C}$ with the following property: for each $s_0 \in \{\sigma > 1\} - \mathbb{R} - S$ that is not a pole of $E^*(w,s)$ or $E^*(w,1-s)$, $\phi(s_0)$ is the only value of $\lambda(s_0) \in \mathbb{C}$ for which $E^*(w,s_0) + \lambda(s_0)E^*(w,1-s_0)$ is an eigenfunction of $-\Delta$ with eigenvalue $s_0(1-s_0)$.

First proof. Let $s_0 \in \{\sigma > 1\} - \mathbb{R}$. That $\lambda(s_0) = \phi(s_0)$ works follows from (5.11). If there are two distinct such λ, then $E^*(w,1-s_0)$ is an eigenfunction. Being the sum of the L^2 functions $\alpha(y)y^{1-s_0}$ and $E^{**}(w,1-s_0)$, it is in L^2. Because $s_0(1-s_0) \notin \mathbb{R}$, this implies $E^*(w,1-s) = 0$. Now should the set of such s_0 be not discrete, then it has an accumulation point, and by uniqueness of analytic continuation we would have $E^*(w,s) = 0$ for all $s \in \mathbb{C}$. In particular, $E(w,s) = 0$. Contradiction.[9] □

Second proof. As before, we know that $\lambda(s_0) = \phi(s_0)$ is a solution, and we want to show that it is the only one. Expressing that $E^*(w,s_0) + \lambda(s_0)E^*(w,1-s_0)$ is an eigenfunction gives:

$$(s_0(1-s_0) - \Delta)E^*(w,s_0) + \lambda(s_0)(s_0(1-s_0) - \Delta)E^*(w,1-s_0) = 0 \tag{5.13}$$

If the coefficient of $\lambda(s_0)$ is nonzero for some w, then $\lambda(s_0)$ is uniquely determined. If it were zero for all w, then $E^*(w,1-s_0)$ is an eigenfunction with eigenvalue $s_0(1-s_0)$, hence it is identically 0. As before, we conclude that the set of such exceptional s_0 is discrete. □

We want to use (5.11) to meromorphically continue $E(w,s)$. We already have a meromorphic continuation of $E^*(w,s)$, so it remains to continue $\phi(s)$. For this, we want to use (5.13).

Proposition 5.14. There is a meromorphic continuation of $\phi(s)$ to the entire complex plane.

Proof. Care must be taken, because it is not obvious that when $\lambda(s_0)$ solves (5.13) for a fixed w, it is a solution for all w. We give an argument that avoids this problem.
We claim that there exists w_0 such that the meromorphic function $(s(1-s) - \Delta)E^*(w_0, 1-s)$ is not identically zero. Indeed: if it were identically zero for all w_0, then $E^*(w,1-s)$ would be a Laplacian eigenfunction, in particular for $\sigma > 1$, where it is L^2. Then E^* is identically zero, a contradiction.
Now, we don't fix s, but we fix w_0 for which $(s(1-s) - \Delta)E^*(w_0, 1-s)$ is a nontrivial meromorphic function. Then (5.13) defines a meromorphic function $\lambda(s)$. It coincides with $\phi(s)$ in the half-plane $\sigma > 1$, because we have seen that there is a dense open subset of that half-plane where there is a unique solution. We conclude that

$$\phi(s) = \frac{(s(1-s) - \Delta)E^*(w_0, s)}{(s(1-s) - \Delta)E^*(w_0, 1-s)}$$

is the desired meromorphic continuation. Note that, a posteriori, $\phi(s)$ does solve (5.13) for all other w_1. Indeed, if $(s(1-s) - \Delta)E^*(w_1, 1-s)$ is not identically zero, then solving (5.13) at $w = w_1$ gives a meromorphic continuation of ϕ, which must be the same as the one we found by evaluating at w_0. If it is identically zero, the equation is trivial. We can also see this as follows: if $\phi(s)$ is constructed by evaluating at some w_0, then $E(w,s) = E^*(w,s) + \phi(s)E^*(w,1-s)$ defines a meromorphic continuation of the Eisenstein series. By joint differentiability, it is still a Laplacian eigenfunction, which implies that (5.13) holds for all w. □

Finally, we conclude, combining (5.10) and (5.14):

Theorem 5.15. The Eisenstein series $E(w,s)$ has a jointly smooth C^∞-meromorphic continuation to the entire complex plane.

More precisely, we have shown that $E(w,s)$ equals $\alpha(y)y^s + \phi(s)\alpha(y)y^{1-s}$ plus an $L^2 \cap C^\infty$-valued meromorphic function.

[9] The Eisenstein series is not identically 0, because it has a nonzero constant term.

5.3 Bernstein's continuation principle

Roughly speaking, Selberg's proof of meromorphic continuation consists of constructing an equation, depending holomorphically on s in a certain sense, satisfied by the Eisenstein series for $\sigma > 1$ and which has a unique solution for all s. Using Fredholm theory, we showed that the unique solution depends meromorphically on s. The proof depends crucially on the specific form of the Fredholm equation. A more general framework is given by Bernstein's *continuation principle*:

(5.16) **Continuation principle.** *Consider a topological $\mathbb{C}$-vector space V, an open connected subset $S \subseteq \mathbb{C}$ and for each $s \in S$ a system of non-homogeneous linear equations for elements $v \in V$, depending holomorphically on $s \in S$. If the system has a unique solution $v(s)$ for all s in some open subset of S, then it has a unique solution for almost all s, which depends meromorphically on s.*

We always assume our topological vector spaces to be Hausdorff. We will make the statement more precise in (5.30). We do not claim that the principle holds always, but we will give a sufficient condition, and apply it to the Eisenstein series.
A continuation principle can also be formulated when S is a higher-dimensional complex manifold. See for example [Garrett, 2001], on which much of the formalization below is based.

5.3.1 Systems of equations

Definition 5.17 (Systems of equations)**.** Let V be a $\mathbb{C}$-vector space. A *(non-homogeneous) linear equation* on V is a triple (T, W, w) where W is a complex vector space, $T : V \to W$ a linear map and $w \in W$. A *system* of linear equations indexed by a set I is a family $\Xi = (T_i, W_i, w_i)_{i \in I}$ of linear equations, and a *solution* of the system is a $v \in V$ such that $T_i v = w_i$ for all $i \in I$. We denote the set of solutions by $\mathrm{Sol}(\Xi)$.

Contrary to [Garrett, 2001], we do not assume V and W to be topological vector spaces and the maps $T : V \to W$ to be continuous. While we will, later, assume that V and W are topological in order to be able to talk about holomorphy, the continuity of the $T : V \to W$ is not necessary for the continuation principle to hold. Although in our application to Eisenstein series, these linear maps will be differential operators on a Fréchet space of smooth functions, and thus indeed continuous.

Remark 5.18.

1. (A system as a single equation) Given a system Ξ, we can always replace it by a single equation without changing the solution set, by taking direct products:
$$\mathrm{Sol}((T_i, W_i, w_i)_{i\in I}) = \mathrm{Sol}\left((T_i)_{i\in I}, \prod_{i\in I} W_i, (w_i)_{i\in I}\right)$$
2. (A system of linear forms) Similarly, we can replace it by a system where all the target spaces have dimension 1. Indeed: take one equation (T, W, w). If $(e_j)_{j\in J}$ is a basis for W and (μ_j) are their dual linear forms (in general not a basis of W^*), then $\mathrm{Sol}(T, W, w) = \mathrm{Sol}((\mu_j \circ T, \mathbb{C}, \mu_j(w))_{j\in J})$. We can then do this for every equation (T, W, w) of the system. More generally, in this argument we can replace the (μ_j) by any family of linear functionals on W that separates points.

To make sense of what it means for a family of systems of equations to be holomorphic, we have to clarify what it means for a function with values in $\mathrm{Hom}(V, W)$ to be holomorphic. Let $U \subseteq \mathbb{C}$ be open. One option is to say that $A : U \to \mathrm{Hom}(V, W)$ is holomorphic iff the $\mathbb{C}$-valued function $s \mapsto \mu(A_s v)$ is holomorphic for all $v \in V$ and $\mu \in W^*$. But this definition is too restrictive: when for example $V = \mathbb{C}$, so that $\mathrm{Hom}(V, W) \equiv W$, W is a topological vector space and $A : U \to W$ is holomorphic in the sense of (B.4), then $\mu \circ A$ is holomorphic when μ is continuous, but usually not fore general μ. It becomes clear that the correct notion of holomorphy must depend on the topology of W.

Definition 5.19. Let V be a $\mathbb{C}$-vector space and W a topological $\mathbb{C}$-vector space. The *weak operator topology* on $\mathrm{Hom}(V, W)$ is the initial topology with respect to the maps $\phi_{v,\mu}$ for $v \in V$ and $\mu \in W^*$ (the continuous dual), defined by

$$\begin{aligned} \phi_{v,\mu} : \mathrm{Hom}(V, W) &\to \mathbb{C} \\ T &\mapsto \mu(T(v)) \end{aligned} \tag{5.20}$$

If for some reason one does not want to restrict to continuous μ, one can always recover that non-topological definition by giving W the discrete topology, so that its algebraic dual and continuous dual coincide.
That it makes $\mathrm{Hom}(V, W)$ into a topological vector space, follows from the general result below applied to $X = \mathrm{Hom}(V, W)$ and $Y = \mathbb{C}$. Alternatively, by noting that this initial topology coincides with the topology induced by the seminorms $p_{v,\mu} = |\phi_{v,\mu}|$. In particular, $\mathrm{Hom}(V, W)$ is a locally convex topological vector space. It is Hausdorff iff the $\phi_{v,\mu}$ separate points of $\mathrm{Hom}(V, W)$. Equivalently, iff W^* separates points.

Proposition 5.21 (Induced topologies on algebraic structures)**.**

1. Let G be a group, H a topological group and $\Phi \subseteq \mathrm{Hom}(G, H)$ a set of homomorphisms. Then G is a topological group for the initial topology with respect to Φ.

2. Let $\mathbb{K}$ be a topological field, X a $\mathbb{K}$-vector space, Y a topological $\mathbb{K}$-vector space and $\Phi \subseteq \mathrm{Hom}(X, Y)$ a set of linear maps. Then X is a topological $\mathbb{K}$-vector space for the initial topology with respect to Φ.

Proof.

1. We show that inversion is continuous on G. It suffices to show that if $U \subseteq G$ is of the form $\phi^{-1}(V)$ with $V \subseteq H$ open, then U^{-1} is open. Because the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \downarrow{\scriptstyle (\cdot)^{-1}} & & \downarrow{\scriptstyle (\cdot)^{-1}} \\ G & \xrightarrow{\phi} & H \end{array}$$

commutes, we have that $U^{-1} = \phi^{-1}(V^{-1})$ is open. We show that the multiplication $G \times G \to G$ is continuous. Let U and V be as before. Because $\phi : G \to H$ is continuous, the map $\phi \times \phi : G \times G \to H \times H$ is continuous. Because the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\phi \times \phi} & H \times H \\ \downarrow{\scriptstyle \bullet} & & \downarrow{\scriptstyle \bullet} \\ G & \xrightarrow{\phi} & H \end{array}$$

commutes, we have that $\bullet^{-1}(U) = (\phi \times \phi)^{-1}(\bullet^{-1}(V))$ is open.

2. Similarly. □

Note that the weak operator topology is usually only defined on the subspace $L(V, W) \subseteq \mathrm{Hom}(V, W)$ of continuous linear maps, when V is also assumed to be a topological vector space.

Definition 5.22 (Holomorphic operator-valued functions)**.** Let $U \subseteq \mathbb{C}$ be open, V and W be $\mathbb{C}$-vector spaces with W topological. We call $A : U \to \mathrm{Hom}(V, W)$ *weakly holomorphic* if the following equivalent conditions hold:

1. A is weakly holomorphic (B.7) for the weak operator topology.

2. All $\mathbb{C}$-valued functions $\phi_{v,\mu} \circ A$ (5.20) are holomorphic.

Proof of equivalence. $1 \implies 2$: Immediate. $2 \implies 1$: By (5.23), every continuous linear functional on $\mathrm{Hom}(V, W)$ is a finite linear combination of the $\phi_{v,\mu}$. □

Proposition 5.23 (The dual of a weak topology)**.** Let V be a $\mathbb{C}$-vector space, whose topology is induced by a family of linear maps $\phi_i : V \to \mathbb{C}$. Then the continuous dual V^* consists of finite linear combinations of the ϕ_i.

Proof. The argument is inspired by the proof of [Bade, 1954, Lemma 3.3]. Let $\theta : V \to \mathbb{C}$ be continuous. By continuity of θ, there exists $\delta > 0$ and a finite set $J \subseteq I$ such that, for all $v \in V$, $|\phi_j(v)| < \delta$ for all $j \in J$ implies $|\theta(v)| < 1$. By linearity, $|\phi_j(v)| < \delta\epsilon$ for all j implies $|\theta(v)| < \epsilon$. In particular, each $\theta(v)$ is determined by the values $\phi_j(v)$. Define $\Phi : V \to \mathbb{C}^{|J|}$ by

$$\Phi(v) = (\phi_j(v))_{j\in J}$$

Then Φ is injective and continuous, and θ factors through Φ. Write $\theta = f_0 \circ \Phi$ for some f_0 defined on the image of Φ. We can extend f_0 to a linear map $f : \mathbb{C}^{|J|} \to \mathbb{C}$. It follows that θ is a linear combination of the ϕ_j. □

Of the plenitude of reasonable topologies[10] on $\mathrm{Hom}(V, W)$, the weak operator topology is the weakest one, so that the corresponding notion of holomorphy is the least restrictive.

Example 5.24. Let V and W be finite-dimensional, with bases (e_j) and (f_k). A family of linear operators $T_s : V \to W$ is holomorphic iff the entries of its matrix representation in these bases are holomorphic.

Definition 5.25 (Holomorphic families of systems of equations)**.** Let V be a $\mathbb{C}$-vector space and $S \subseteq \mathbb{C}$ open.

1. A holomorphic family of linear equations on V is a family $(T_s, W, w_s)_{s\in S}$ where W is a topological $\mathbb{C}$-vector space, $T : S \to \mathrm{Hom}(V, W)$ is weakly holomorphic and $w : S \to W$ is weakly holomorphic.
2. A holomorphic family of systems of equations is a system of holomorphic families of equations:

$$\Xi(s) = ((T_s^{(i)}, W^{(i)}, w_s^{(i)})_{s\in S})_{i\in I}$$

For convenience, we will call it simply a *holomorphic system*. The solution set becomes a function of s:

$$\mathrm{Sol}(\Xi(s)) = \mathrm{Sol}((T_s^{(i)}, W_s^{(i)}, w_s^{(i)})_{i\in I})$$

Remark 5.26 (Holomorphic systems of linear forms)**.** Suppose that the continuous dual W^* separates points, which is for example the case when W is Hausdorff and locally convex, by the Hahn–Banach separation theorem. Let (μ_i) be any generating set of W^*. By Remark 5.18, an equation (T, W, w) is equivalent to the system $(\mu^{(i)} \circ T, \mathbb{C}, \mu^{(i)}(w))_{i\in I}$. We also have that a holomorphic family of equations (T_s, W, w_s) is holomorphic iff $(\mu^{(i)} \circ T_s, \mathbb{C}, \mu^{(i)}(w_s))_i$ is holomorphic: for the inhomogeneous terms w_s this follows from the definition of weak holomorphy. For the linear maps, similarly.

Proposition 5.27 (Composition of holomorphic families)**.** Let U, V, W be $\mathbb{C}$-vector spaces with V and W topological. Let $S \subseteq \mathbb{C}$ be open and $A : S \to \mathrm{Hom}(U, V)$ and $B : S \to L(V, W)$ weakly holomorphic.

1. Then $B \circ A : s \mapsto B_s \circ A_s$ is holomorphic.
2. If $v : S \to V$ is weakly holomorphic, then $Bv : s \mapsto B_s v_s$ is weakly holomorphic.

Proof. [Garrett, 2001]

1. This is a corollary of Hartog's theorem on separate analyticity. Let $\mu \in W^*$ and $u \in U$. By definition of weak holomorphy and continuity of the B_t, the $\mathbb{C}$-valued function

$$\begin{aligned} S \times S &\to \mathbb{C} \\ (s, t) &\mapsto \mu(B_t(A_s(u)) \end{aligned}$$

is separately analytic. By Hartog's theorem, it is jointly analytic. In particular, the diagonal function

$$s \mapsto (s, s) \mapsto \mu(B_s(A_s(u))$$

is analytic.

[10] strong, weak, ultrastrong, σ-weak, ...

2. With the identification $V \equiv \mathrm{Hom}(\mathbb{C}, V)$, a weakly holomorphic v_s corresponds to a weakly holomorphic operator-valued function. Thus this reduces to the first statement. □

Remark 5.28. In the setting of (5.27), note that the map $S \to \mathrm{Hom}(U,V) \oplus \mathrm{Hom}(V,W) : s \mapsto (A_s, B_s)$ is weakly holomorphic. One could try to give an elementary argument of the first statement by showing that composition of linear maps $\mathrm{Hom}(U,V) \oplus \mathrm{Hom}(V,W) \to \mathrm{Hom}(U,W)$ is holomorphic (in the sense of (B.4)) for the weak operator topologies, and it will follow that the composition

$$\begin{array}{ccccc} S & \longrightarrow & \mathrm{Hom}(U,V) \oplus \mathrm{Hom}(V,W) & \longrightarrow & \mathrm{Hom}(U,W) \\ s & \longmapsto & (A_s, B_s) & \longmapsto & B_s \circ A_s \end{array}$$

is holomorphic. But composition of linear operators is in general not holomorphic for the weak operator topology; it need not even be continuous.

Finally, we clarify what it means for a V-valued function to be meromorphic.

Definition 5.29. Let V be a topological $\mathbb{C}$-vector space and $U \subseteq \mathbb{C}$ open. A function $f : U \to V$ is (weakly) *meromorphic* if it is locally of the form $\frac{g}{P}$ with $g : U \to V$ (weakly) holomorphic (B.4) and P a $\mathbb{C}$-valued polynomial. Note that in the case of Fréchet spaces (or whenever Laurent expansions exist), this corresponds to the usual definition in terms of Laurent coefficients (Appendix B.4).

We are ready to formulate a precise continuation principle:

(5.30) **Continuation principle.** *Let V be a topological $\mathbb{C}$-vector space, $S \subseteq \mathbb{C}$ be open and connected, and $\Xi(s)_{s\in S}$ a holomorphic system of equations on V, in the sense of (5.25). Suppose there is an open subset $U \subseteq S$ such that for $s \in U$, the system has a unique solution $v(s)$. Then $\Xi(s)$ has a unique solution for all $s \in S$ except possibly a closed discrete set, and the solution $v(s)$ is weakly meromorphic in s. In particular, $v(s)$ is meromorphic in U.*

5.3.2 Systems of finite type

The prime example of holomorphic systems for which the continuation principle holds, are systems which "essentially" consist of only finitely many equations.

Definition 5.31 (Systems of finite type)**.** Let V be a topological $\mathbb{C}$-vector space and $\Xi(s)_{s\in S}$ be a holomorphic system of equations on V.

1. We say $\Xi(s)$ is of *finite type* if the following equivalent conditions hold:
   (a) There is a holomorphic family of vector subspaces of bounded dimension containing $\mathrm{Sol}(\Xi(s))$ for all s. Formally: there exists a finite-dimensional vector space L and a holomorphic family of linear maps $\lambda : S \to \mathrm{Hom}(L, V)$ such that $\mathrm{Sol}(\Xi(s)) \subseteq \lambda_s(L)$ for all $s \in S$. We call $(L, (\lambda_s)_{s\in S})$ a *finite holomorphic envelope.*
   (b) There is a holomorphic family of *affine* subspaces of bounded dimension containing $\mathrm{Sol}(\Xi(s))$ for all s. Formally: there exists a finite-dimensional vector space L, a holomorphic family of linear maps $\mu : S \to \mathrm{Hom}(L, V)$ and a holomorphic map $v : S \to V$ such that $\mathrm{Sol}(\Xi(s)) \subseteq \mu_s(L) + v_s$ for all $s \in S$.
2. We say $\Xi(s)$ is *locally of finite type* if every $s \in S$ has an open neighborhood on which Ξ is of finite type.

Proof of equivalence. If (a) holds, then (b) follows with $\mu = \lambda$ and $v_s = 0$ for all s. If (b) holds, then (a) follows with $L' = L \oplus \mathbb{C}$ and $\lambda_s(l, z) = \mu_s(l) + z \cdot v_s$. □

Example 5.32. 1. If V is finite-dimensional, every system is of finite type.

2. If $\Xi(s)$ has a unique solution $v(s)$ for all s which depends holomorphically on s, then Ξ is of finite type: We can take $L = \{0\}$ in condition (b).

Recall that a finite-dimensional $\mathbb{C}$-vector space L has a canonical topology: it is the topology induced by any norm. It comes with a canonical notion of L-valued holomorphic functions. Recall also that a holomorphic system can often be reduced to a holomorphic system whose equations have target spaces that are one-dimensional (5.26).

Theorem 5.33 (Continuation principle for finite type systems)**.** With the notations from the continuation principle (5.30): if $\Xi(s)$ is of finite type and consists of equations whose target spaces W_i are one-dimensional, then the continuation principle holds.
If moreover V is locally convex and quasi-complete, so that V-valued (weakly) meromorphic functions have Laurent-expansions (Appendix B.4), then $v(s)$ is holomorphic in U.

The idea is that the unique solvability of a finite system of equations can be expressed in terms of the nonvanishsing of a determinant. If the determinant is a holomorphic function that does not vanish in some open set, then it vanishes globally at at most a discrete set of points. There is a subtlety: the determinant will be the one of the linear system $T_s \circ \lambda_s$ on L, but this system has zero determinant if λ_s is not injective:

Proof. Let $\lambda_s : L \to V$ be a finite holomorphic envelope of $\Xi(s)$.
Injective envelope. Suppose first that λ_s is injective for all s. We will later remove this hypothesis. Because $\mathrm{Sol}(\Xi(s)) \subset \lambda_s(L)$ we have the equality $\mathrm{Sol}(\Xi(s)) = \lambda_s\left(\mathrm{Sol}(\Xi(s) \circ \lambda_s)\right)$. If $\Sigma(s) = \Xi(s) \circ \lambda_s$ has a unique solution $u(s)$ for all s, then $\Xi(s)$ has the unique solution $v(s) = \lambda_s(u(s))$. And if $u(s)$ is weakly meromorphic, then so is $v(s)$ (5.27).
Reduction to a finite system. Hence, replacing, $\Xi(s)$ by $\Xi(s) \circ \lambda_s$, which is still a holomorphic system by (5.27), we may suppose that $V = L$ is finite-dimensional and that $\lambda_s = \mathrm{id}_V$ for all s. We may also suppose $V = \mathbb{C}^N$. Let (e_j) be the standard basis of V, and denote by (x_j) the coordinates of $x \in V$. By assumption, Ξ consists of equations of the form

$$\sum_{j=1}^{N} a_{ij}(s) x_j = w_i(s) \qquad (i \in I)$$

where the a_{ij} are holomorphic by (5.24).
Now let $U \subseteq S$ be open such that $\Xi(s)$ has a unique solution $v(s)$ for $s \in U$. Take any $s_0 \in U$. Select N equations of $\Xi(s)$ that determine the solution for $s = s_0$, say the equations corresponding to $i_1, \ldots, i_N$. Then the determinant $\det(a_{i_k j}(s_0)) \neq 0$. This determinant is a holomorphic function of s, hence $\det(a_{i_k j}(s)) \neq 0$ for all $s \in S$ except possibly for a closed discrete set P. For such s, by Cramer's rule the unique solution of the subsystem is given in terms of the adjugate matrix by

$$u(s) = \frac{1}{\det(a_{i_k j}(s))} \operatorname{adj}(a_{i_k j}) \begin{pmatrix} w_{i_1}(s) \\ \vdots \\ w_{i_N}(s) \end{pmatrix}$$

In particular, u is a weakly meromorphic V-valued function in the sense of (5.29). Because $u(s)$ is the unique solution of a subsystem, for $s \in U - P$ it must coincide with the solution $v(s)$ of the full system, which is unique by assumption. In particular, $v(s)$ is weakly holomorphic on $U - P$. By repeating the argument with s_0 a point of $P \cap U$, we conclude that $v(s)$ is weakly holomorphic on U.
Solvability of the full system. We did not show that the meromorphic continuation $u(s)$ is a solution of the full system for $s \in S - P$. It is true: $u(s)$ being a solution of a linear equation of $\Xi(s)$ is a holomorphic condition in s. It is satisfied in U, hence by uniqueness of analytic continuation for $\mathbb{C}$-valued functions, it is satisfied in $S - P$.
A solution of $\Xi(s)$ for $s \notin S - P$ (which exists, as we now know) is automatically unique: already the solution of a finite subsystem is unique.
Reduction to injective λ_s. Finally, we get rid of the assumption that the λ_s are injective. The dimension of the range of λ_s is bounded by $\dim L$. Take $s_0 \in S$ for which the dimension is maximal,

and take a subspace $L' \subseteq L$ of minimal dimension such that $\lambda_{s_0}(L') = \lambda_{s_0}(L)$. Then in particular $\lambda_{s_0}|_{L'}$ is injective. Because λ_s is holomorphic, $\lambda_s|_{L'}$ is injective for $s \in S - R$, for some closed and discrete $R \subseteq S$. In particular, $\lambda_s(L') = \lambda_s(L)$ for such s.
We now apply the case where "$\lambda_s = \mathrm{id}_V$" to the system $\Xi(s) \circ \lambda_s$. We know that this has a unique solution for $s \in U - R$, say $v(s)$, it is weakly holomorphic in $U - R$ and it has a weakly meromorphic continuation to S. (It may have poles outside R.) Let $w(s)$ be the unique solution of $\Xi(s)$ for $s \in U$. By uniqueness, we must have $w(s) = \lambda_s(v(s))$ for $s \in U - R$, and we conclude that $\lambda_s(v(s))$ is a weakly meromorphic continuation of $w(s)$ in S, but its values may differ from those of w at points of $U \cap R$. This completes the proof of the continuation principle.
Holomorphy in U. We would like to show that:[11]

- $\lambda_s(v(s))$ has removable singularities in U.
- It coincides with w at points of $U \cap R$.

Call $y(s) = \lambda_s(v(s))$, we only look at it for $s \in U - R$. Take $s_0 \in R \cap U$. By assumption, $\Xi(s)$ is of finite type in S, and because we now know that it has a meromorphic solution, we can show that it is of finite type with injective envelope in a neighborhood of s_0. Indeed: suppose s_0 is a pole of order $N > 0$ of $y(s)$ and take a small neighborhood Ω of s_0 in which there are no other points of R. Let $L'' = \mathbb{C}^2$ and define $\mu_s : L'' \to V$ for $s \in \Omega$ by

$$\mu_s(a, b) = y(s)(s - s_0)^N a + w(s_0)b$$

Where we extend $y(s)(s - s_0)^N$ holomorphically to s_0; call its value y_0. Suppose y_0 and $w(s_0)$ are linearly independent. Then the above μ_s provides a finite holomorphic envelope of $\Xi(s)$ for $s \in \Omega$, and moreover μ_{s_0} is injective. By the injective case, we deduce that $y(s)$ is holomorphic in a neighborhood of s_0, in particular, at s_0. This contradicts $N > 0$. Now suppose y_0 and $w(s_0)$ are linearly dependent. Then

$$\kappa_s(a) = y(s)(s - s_0)^N a$$

defines a one-dimensional holomorphic envelope of $\Xi(s)$ for s around s_0. Suppose V has Laurent-expansions. Then κ_{s_0} is injective: $y_0 = 0$ would imply that s_0 is a pole of $y(s)$ of smaller order. By the injective case, we conclude again that y is weakly holomorphic at s_0. (A contradiction with $N > 0$.) □

Holomorphy is a local condition, so not surprisingly, we obtain:

Theorem 5.34 (Continuation principle for locally finite type systems)**.** If $\Xi(s)$ is *locally* of finite type and consists of equations whose target spaces W_i are one-dimensional, then the continuation principle (5.30) holds for $\Xi(s)$.

Proof. Write S as an increasing union $(U_n)_{n \geqslant 1}$ of open relatively compact (in S) open sets. On each of them, there is a weakly meromorphic continuation $v_n(s)$ by the locally finite case. We show that they glue together to a weakly meromorphic $v(s)$. Let P be the set of points at which $\Xi(s)$ does not have a unique solution. By the finite type case, P is closed and discrete in each of the U_n. Hence it is closed and discrete in their union, S. By uniqueness of the solution, the v_n are successive extensions on $U_n \cap (S - P)$, and it follows that when we glue them together to a weakly holomorphic v on $S - P$, this $v(s)$ is weakly meromorphic. □

5.3.3 Criteria for finiteness

We need some practical criteria to assure that a holomorphic system is locally of finite type.

Proposition 5.35 (Dominance)**.** (Called *inference* by Bernstein) [Garrett, 2014a, Proposition 2.0.6] Let V and V' be topological $\mathbb{C}$-vector spaces, $U \subseteq \mathbb{C}$ open and Ξ and Ξ' holomorphic systems on V

[11] This is automatic if we know on beforehand that $w(s)$ is holomorphic in U, which will be the case when we apply the uniqueness principle to the Eisenstein series.

and V', respectively. We say that X' *dominates* X if there exists a weakly holomorphic family (for the weak operator topology) of continuous linear maps $h_s : V' \to V$ such that

$$\operatorname{Sol}\Xi(s) \subseteq h_s(\operatorname{Sol}\Xi'(s)) \quad (\forall s \in U) \tag{5.36}$$

If Ξ' is (locally) finite, then Ξ is (locally) finite.

Proof. If (L, λ_s) is a finite holomorphic envelope for Ξ' in some open subset of U, then $h_s \circ \lambda_s$ is one for Ξ. It is holomorphic by (5.27). □

Note that if $\Xi'(s) = \Xi(s) \circ h_s$, then the reverse inclusion in (5.36) holds.

Proposition 5.37 (Compact operator criterion)**.** [Garrett, 2014a, Proposition 2.0.7] Let V be a $\mathbb{C}$-Hilbert space, $U \subseteq \mathbb{C}$ open and Ξ be the system on V determined by a holomorphic family of bounded operators $T_s : V \to V$, holomorphic for the norm topology on $L(V, V)$. Suppose that for some $s_0 \in U$, T_{s_0} is of the form $\lambda + K$ with $\lambda \in \mathbb{C} - \{0\}$ and K compact. Then Ξ is of finite type in a neighborhood of s_0.

Proof. The kernel of T_{s_0} is closed because T_{s_0} is bounded, and its range is closed because K is compact: this is part of the spectral theory of compact operators (usually only formulated for operators from a Banach space to itself). Call V_0 its kernel and V_1 its range, and let pr_i be the orthogonal projection on V_i. The system $\operatorname{pr}_1 \circ\Xi$ dominates Ξ. We how that it is of finite type around s_0. Define

$$\phi_s = \operatorname{pr}_0 \oplus (\operatorname{pr}_1 \circ T_s) : V \to V_0 \oplus V_1$$

and note that $\operatorname{Sol}(\operatorname{pr}_1 \circ\Xi(s)) = \phi_s^{-1}(V_0 \oplus 0)$. By spectral theory of compact operators, V_0 is finite-dimensional, so it suffices to show that ϕ_s is invertible and that its inverse is holomorphic.
By construction, ϕ_{s_0} is a continuous linear bijection, and it is holomorphic in s. By the open mapping theorem (A.13), it is an isomorphism. Because ϕ_s is holomorphic for the norm topology, it is in particular continuous, so that ϕ_s is invertible for s in a neighborhood of s_0, and the inverse is also holomorphic (B.6). We conclude that $\operatorname{pr}_1 \circ\Xi$, and thus Ξ, is of finite type around s_0. □

With minor modifications, a similar criterion can be proven more generally for Fredholm operators (bounded operators that are invertible modulo compact operators) between Banach spaces, but we will not need it. See again [Garrett, 2014a, Proposition 2.0.7]. Instead, we will be interested in the following generalization which is suitable for non-homogeneous systems:

Proposition 5.38 (Compact operator criterion, inhomogeneous version)**.** Let $V, U, T_s, \lambda, K, T_{s_0}$ be as before. Let L be finite-dimensional and let $w_s : L \to V$ be bounded operator-valued, holomorphic for the operator norm. Then there exists a finite holomorphic envelope (L', μ_s) in a neighborhood Ω of s_0 with

$$T_s^{-1}(w_s(L)) \subseteq \mu_s(L') \qquad \forall s \in \Omega$$

Proof. The argument is the same: we prove the stronger result that $T_s^{-1}\operatorname{pr}_1^{-1}(\operatorname{pr}_1(w_s(L)))$ has a finite holomorphic envelope around s_0, by observing that

$$T_s^{-1}\operatorname{pr}_1^{-1}(\operatorname{pr}_1(w_s(L))) = \phi_s^{-1}((0, w_s(L)) + (V_0 \oplus 0))$$

So we can take $L' = L \oplus V_0$. □

5.3.4 Eisenstein series

We want to apply the continuation principle to the Eisenstein series, so we're looking for a holomorphic system of locally finite type that characterizes the Eisenstein series. By the uniqueness principle (4.44), we have that for $\sigma > 1$ the Eisenstein series $f = E(\cdot, s)$ is characterized by

$$\begin{cases} (\Delta + s(1-s))f = 0 \\ C_f = y^s + \phi(s)y^{1-s} \text{ for some } \phi(s) \in \mathbb{C} \end{cases}$$

Equivalently, by the system of linear equations

$$\begin{cases} (\Delta + s(1-s))f = 0 \\ \left(y\frac{\partial}{\partial y} - (1-s)\right)(C_f - y^s) = 0 \end{cases} \tag{5.39}$$

Consider the Fréchet space of smooth functions $V = C^\infty(\Gamma\backslash\mathbf{H})$ and the vector subspace W of functions that are of polynomial growth. The system (5.39) is linear (inhomogeneous) in $f \in W$, and is holomorphic (using (B.23)). With the aim of applying the continuation principle, we want this system to be locally of finite type, and the continuation principle (5.34) will apply. Indeed: The equations of the system are W-valued, and because V is Fréchet, continuous linear functionals of V (hence of W) separate points, so that the system can be reduced to an equivalent holomorphic system with equations whose target spaces are one-dimensional (5.26).
Note that the set of solutions of (5.39) of polynomial growth is finite for all s. Indeed, already without the second equation this is true: this is precisely saying that spaces of Maass forms are finite-dimensional (4.46).
In unpublished lecture notes [Bernstein, 1984, Lecture III, §5], one finds the informal remark that, if one can prove that a holomorphic system has finitely many solutions for all s, usually one can also prove that it is locally of finite type. A more detailed argument is supposedly given in an appendix to these lecture notes, but we haven't found this appendix, and we give an alternative to this rather imposing remark. It should be noted that verifying local finiteness is something that has often been neglected in existing proofs of meromorphic continuation that use the continuation principle [Garrett, 2012a].

Proposition 5.40. The holomorphic system on W defined by the equation $(\Delta + s(1-s))f = 0$ is locally of finite type for the L^2_{loc}-topology on W.

It suffices to prove the following strengthening of finite-dimensionality of the spaces of Maass forms $H(\Gamma, \lambda)$:

Proposition 5.41. For every $s_0 \in \mathbb{C}$, there exists an open neighborhood U of s_0, a finite-dimensional vector space L and a strongly holomorphic family of linear maps $\lambda_s : L \to W$ $(s \in U)$ such that

$$\lambda_s(L) \supseteq H(\Gamma, s(1-s)) \qquad (\forall s \in U)$$

Here W is equipped with the L^2_{loc}-topology.

Proof. We take a second look at proof 4 of (4.46). We know that Maass forms in $H(\Gamma, s(1-s))$ are eigenfunctions of an automorphic kernel K, with eigenvalue $\widehat{k}(s)$. We want to reduce to the compact operator criterion (5.38) and work with L^2-functions, so we truncate our Maass forms: we know that each $f \in H(\Gamma, s(1-s))$ has a constant term of the form $ay^s + by^{1-s}$. Let $A > 0$ be large enough such that there are no elliptic points with imaginary part $\geqslant A$, so that there exists $\alpha \in C^\infty(\mathbb{R})$ of the form

$$\alpha(y) = \begin{cases} 0 & : y \leqslant A \\ 1 & : y \geqslant A+1 \end{cases}$$

which defines a smooth function on $\Gamma\backslash\mathbf{H}$, as in the section about truncated kernels. Then $f \in \mathbb{C}\cdot\alpha(y)y^s + \mathbb{C}\cdot\alpha(y)y^{1-s} + L^2(\Gamma\backslash\mathbf{H})$. Define $\operatorname{trunc} f := f - \alpha(y)C_f$. The functions $\alpha(y)y^s$ and $\alpha(y)y^{1-s}$ define a C^∞-holomorphic envelope (L, λ_s) for $f - \operatorname{trunc} f$, with $L = \mathbb{C}^2$. It remains to find an envelope for $\operatorname{trunc} f$. Applying the approximate Selberg eigenfunction principle to the truncated kernel L_3 (4.37) gives

$$(L_3 - \widehat{k}(s))\operatorname{trunc} f = L_3(\alpha(y)C_f)$$

(Compare with (5.8).) That is, the RHS lies in the finite-dimensional subspace of L^2 spanned by $L_3(\alpha(y)y^s)$ and $L_3(\alpha(y)y^{1-s})$. These functions are (w,s)-continuous and pointwise holomorphic with compact support bounded uniformly in s, so they are L^2-holomorphic (B.27). They give a finite envelope for $(L_3 - \widehat{k}(s))\operatorname{trunc} f$. Take k to be an approximation of the identity, so that $\widehat{k}(s_0) \to 1$

(3.34). Using the compact operator criterion, we conclude that $\operatorname{trunc} f$ takes values in some L^2-holomorphic finite envelope (L', μ_s), locally around s_0.
We want to combine both envelopes. But λ_s is holomorphic for the Fréchet topology, while μ_s is holomorphic for the L^2-topology. So all we can say for now is that the local envelope $(L \oplus L', \lambda_s \oplus \mu_s)$ for $H(\Gamma, s(1-s))$ is holomorphic for the L^2_{loc} topology. □

Corollary 5.42. The Eisenstein series has a W-valued weakly L^2_{loc}-meromorphic continuation to $\mathbb{C}$.

Proof. This follows now from the continuation principle (5.34). Note that L^2_{loc}-continuous functionals of W still separate points: one can take the convolution of L^2_{loc} functions with compactly supported 'test functions'. □

Note that while the L^2_{loc}-topology is weaker than the C^∞-Fréchet topology, the L^2_{loc}-topology is still Fréchet; its topology is induced by a countable family of seminorms. In particular, the weak-to-strong principle for holomorphy (hence meromorphy) holds for L^2_{loc} (B.8):

Corollary 5.43. The Eisenstein series has a W-valued strongly L^2_{loc}-meromorphic continuation to $\mathbb{C}$.

The L^2_{loc}-topology on W is so weak that we cannot even conclude that $E(w, s)$ is meromorphic for fixed w: evaluation maps are continuous for the usual Fréchet-topology, but not for the L^2_{loc}-topology. So proving L^2_{loc}-meromorphy hardly counts as proving a meromorphic continuation. We try to upgrade to C^∞-meromorphy.
We want to use an L^2-to-C^∞ result for vector-valued meromorphic functions (B.31). The only issue is joint continuity of $E(w, s)$ and its partial derivatives w.r.t. w, away from poles. All we know is that they are separately continuous and L^2_{loc}-continuous. But they are harmonic for fixed w and a Laplacian eigenfunction for fixed s, so we are in a setting similar to Hartog's theorem on separate holomorphy:

Proposition 5.44 (Separate eigenfunctions are jointly smooth)**.** Let $f : \mathbf{H} \times U \to \mathbb{C}$ be (jointly) locally integrable, separately smooth, harmonic for fixed $w \in \mathbf{H}$ and annihilated by $\Delta + s(1-s)$ for fixed $s \in U$. Then f is jointly smooth, after changing its values on a set of measure 0.

Proof. This follows from elliptic regularity for overdetermined systems of differential equations (F.26). If $\Delta_{\mathbf{H}}$ and Δ_U denote the respective Laplace operators, then f is annihilated in the distributional sense by $\Delta_{\mathbf{H},w} - s(1-s)$ and $\Delta_{U,s}$: For every test function $\phi \in C_c^\infty(\mathbf{H} \times U)$,

$$\begin{aligned}\int_{\mathbf{H}} \int_U f(w,s) \Delta_{U,s} \phi(w,s) ds dw &= \int_{\mathbf{H}} \int_U \Delta_{U,s} f(w,s) \phi(w,s) ds dw \\ &= \int_{\mathbf{H}} 0 \\ &= 0\end{aligned}$$

and similarly for $\Delta_{\mathbf{H}}$. These differential operators have principal symbols $y^2(\xi_1^2 + \xi_2^2)$ resp. $\xi_3^2 + \xi_4^2$, which do not vanish simultaneously for $(\xi_i) \in \mathbb{R}^4 - \{0\}$, so they constitute an overdetermined elliptic system. □

One can show that a separately continuous function $\mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$ is always Lebesgue-measurable [Johnson, 1969].
We are almost in a position to prove that $E(w, s)$ has a W-valued strongly C^∞-meromorphic continuation. By the above discussion, it remains to show that $E(w, s)$ is jointly smooth and jointly locally integrable. We know that it is a continuous L^2_{loc}-valued function, so in particular its local L^2-norms are locally bounded when s varies. By Fubini, it follows that $E(w, s)$ is jointly L^2_{loc}.
Joint smoothness is more subtle: what (5.44) tells us is that $E(w, s)$ is jointly smooth *after changing its values on a set of measure* 0. Call $F(w, s)$ the smooth modification of $E(w, s)$. We want the difference $F(w, s) - E(w, s)$ to be 0. All way know is:

- It is almost everywhere 0.
- It is separately smooth.

While it is true that continuous functions which are zero almost everywhere are zero everywhere, it is a priori unclear whether the same conclusion holds for separately continuous functions. If this is true, then it will follow that $F(w, s) - E(w, s)$ is everywhere 0, that $E(w, s)$ is smooth and finally, using (B.31), that its L^2_{loc}-meromorphy implies C^∞-meromorphy. We don't know how to fill this gap.

5.4 Further analysis

So far we have proved a C^∞-meromorphic continuation of $E(w, s)$, jointly smooth away from poles. Much more can be said about the Eisenstein series:

Theorem 5.45. We have the functional equations

$$E(w, 1-s) = \phi(s)E(w, s)$$

$$\phi(s)\phi(1-s) = 1$$

Proof. Using the uniqueness principle (4.44), this is almost immediate: both sides are Maass forms whose constant terms are of the form $y^{1-s} + \phi(1-s)y^s$ resp. $\phi(s)y^s + \phi(s)\phi(1-s)y^{1-s}$. For $\sigma > 1$, the uniqueness principle implies the first functional equation. It extends to general s by uniqueness of meromorphic continuation. Comparing their constant terms, we obtain the functional equation for ϕ. □

Note: it is not obvious why (whether) uniqueness of meromorphic continuation holds for vector-valued meromorphic functions. But because the C^∞-Fréchet topology on $V = C^\infty(\mathbf{H})$ separates points (already evaluations separate points) uniqueness of meromorphic continuation on V reduces to uniqueness of meromorphic continuation for $\mathbb{C}$-valued functions.
In (5.4), we showed that the meromorphic continuation of $E(w, s)$ has no poles in the set $\{\sigma > \frac{1}{2}\} - [0, 1]$. Using the so-called *Maass–Selberg relations* for L^2-inner products of truncated Eisenstein series, one can show:

Theorem 5.46. Poles can only occur in the half-open interval $(0, 1]$, and they are simple. The residues are square-integrable Maass forms.

Proof. See e.g. [Borel, 1997, §12.11]. □

Theorem 5.47. $s = 1$ is a pole, with residue equal to the constant function 1.

Proof. See e.g. [Borel, 1997, Proposition 12.13]. □

A Functional analysis

In this entire section, $\mathbb{K}$ will denote any of the two fields $\mathbb{R}$ and $\mathbb{C}$.

Definition A.1. Let V be a normed space. A *Schauder basis* for V is a family (s_i) for which every element can be written uniquely as a convergent series $\sum_{i \in I} \lambda_i s_i$ (where only countable many terms are nonzero).

A permutation of a Schauder basis need no longer be Schauder. In a Hilbert space, it does, and the series $\sum_{i \in I} \lambda_i s_i$ converges absolutely whenever it converges. (Bessel's inequality)

Definition A.2. A Hilbert space over $\mathbb{K}$ is *separable* if the following equivalent conditions hold:

- It is separable as a topological space, i.e. has a countable dense subset.
- It has a countable Schauder basis.
- It has a countable orthonormal Schauder basis.

It implies that every Schauder basis is countable. 'Basis' will mean Schauder basis from now on.

Example A.3. For $p \geqslant 1$ the space of sequences $\ell^p(\mathbb{K})$ is Banach. For X a measure space, $L^p(X)$ is a Banach space (Riesz-Fischer). For $p = 2$ they are Hilbert.

Example A.4. For a set X and a Banach space V, the bounded functions $X \to V$ with the supremum norm $\|\cdot\|_\infty$ form a Banach space. If X has a topology and we restrict to continuous maps, the resulting space is still complete.

A.1 Bounded operators

Definition A.5 (Bounded operator)**.** Let X and Y be normed spaces. A linear transformation $A : X \to Y$ is *bounded* if the following equivalent conditions hold:

- It is continuous.
- The preimage of the unit ball of Y contains an open ball around $0 \in X$.
- The image of the unit ball of X is bounded.
- The image of a bounded set is bounded.
- $\|A(x)\| \ll \|x\|$ uniformly for $x \in X$.

in which case we can define its norm $\|A\|$ as

$$\sup_{x \in X} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| = \inf\{C > 0 : \|Ax\| \leqslant C\|x\| \, \forall x \in X\}$$

Example A.6. In an inner product space X over $\mathbb{R}$ or $\mathbb{C}$, for each $x \in X$ the linear map $\langle x, \cdot \rangle : X \to \mathbb{C}$ is bounded by Cauchy-Schwarz, with norm equal to $\|x\|$.

Example A.7 (Integral operators)**.** Let X be a measure space and $k \in L^2(X \times X)$ be a $\mathbb{C}$-valued 'kernel'. Then the operator K on the normed space $L^2(X)$ defined by

$$f \mapsto \int k(s,t) f(t) dt$$

is bounded with norm $\leqslant \|K\|_2 = \int_{X^2} |k|^2$.

We will see that it is compact (A.33) and even Hilbert–Schmidt (A.56).

Example A.8. Let X, Y be normed spaces over $\mathbb{K}$. The bounded linear operators $L(X, Y)$ form again a normed space for the operator norm. In particular we can consider the *dual* $X^* = L(X, \mathbb{K})$.

We state some useful facts about bounded operators:

Proposition A.9. Let X, Y be Banach spaces over $\mathbb{K}$. The bounded linear operators $L(X, Y)$ form a Banach space.

Theorem A.10 (Hahn-Banach)**.** Let X be a Banach space over $\mathbb{K}$ and $Y \leqslant X$ a linear subspace. For every bounded linear map $\phi_0 \in L(Y, \mathbb{K})$ there is a bounded linear extension $\phi \in L(X, \mathbb{K})$ with the same norm.

Corollary A.11. Let X be a Banach space over $\mathbb{K}$ and $x \in X$ be nonzero, then there exists $\phi \in L(X, \mathbb{K})$ with $\phi(x) = \|x\|$ and with $\|\phi\| = 1$.

Proof. Apply Hahn-Banach to the linear subspace $Y = \mathbb{K}x \subseteq X$ and the bounded linear map $\phi_0(\lambda x) = \lambda / \|x\|$ which has norm 1. ☐

Proposition A.12. Let $A : X \to Y$ be a linear map between separable inner product spaces. Let (a_{ij}) be the (infinite) matrix of A in orthonormal bases. Then for A to be bounded it suffices that $\sum_{i,j} |a_{ij}|^2 < \infty$.

This condition is strong (it says that A is Hilbert–Schmidt). If the matrix is diagonal (in particular, if $X = Y$ and A is diagonalizable), it suffices that its entries are bounded (A.42). Thus the change of orthonormal bases defines a bounded operator (by continuous extension of a densely defined operator, or Hahn-Banach).

Theorem A.13 (Open mapping theorem)**.** Let $A : X \to Y$ be a surjective bounded map between Banach spaces. Then A is open.

Theorem A.14 (Closed graph theorem)**.** Let $A : X \to Y$ be a linear map between Banach spaces. Then A is bounded iff its graph $G \subset X \times Y$ is closed.

Proof. If A is bounded, the graph is closed by continuity. (This is true for any continuous map to a Hausdorff space.) Conversely, if G is closed, it is a Banach subspace of $X \times Y$. The first projection $G \to X$ is a continuous bijection, and by the open mapping theorem it is an isomorphism. Thus $X \to G : x \mapsto (x, Ax)$ is continuous, hence so is A. ☐

A.2 The adjoint of an operator, C^*-algebras

Theorem A.15 (Riesz representation theorem)**.** On a Hilbert space, every bounded linear form is of the form $\langle x, \cdot \rangle$ for a unique x.

Definition A.16. Let $A : X \to Y$ be a bounded operator between Hilbert spaces. Its adjoint is the unique mapping $A^* : Y \to X$ such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x \in X$, $y \in Y$.

Proposition A.17 (Properties of the adjoint)**.** We have that A^* is linear, is bounded with $\|A\| = \|A^*\|$, $(\lambda A)^* = \bar{\lambda} A^*$ for $\lambda \in \mathbb{C}$, $A^{**} = A$ and $(AB)^* = B^* A^*$ whenever this makes sense. If $X = Y$ then $\|AA^*\| = \|A^2\|$.

Definition A.18. A Banach-algebra over $\mathbb{K}$ is a complete, unital, associative normed $\mathbb{K}$-algebra, 'normed' meaning that the norm is submultiplicative:

$$\|xy\| \leqslant \|x\| \|y\| \qquad \forall x, y \in A$$

A C^*-algebra is a complex Banach-algebra with an involution (adjoint) satisfying the properties in the above proposition.

While it can be of interest to study non-unital C^*-algebras, we only need to know about the algebra of operators on a Hilbert-space, so we require our C^*-algebras to be unital.

Definition A.19. Let A be a C^* algebra and $x \in A$. Then x is normal if $xx^* = x^* x$, and self-adjoint if $x = x^*$.

On a complex Hilbert space, a bounded operator is self-adjoint iff $\langle Ax, x \rangle \in \mathbb{R}$ for all x.

A.3 Banach-Alaoglu and the Gelfand-transform

Theorem A.20 (Banach-Alaoglu)**.** Let X be a normed space over $\mathbb{K}$ with dual X^*. As a topological space we can view X^* as a subspace of the product $\mathbb{K}^X$. Then with the topology induced by the product topology, the closed unit ball B^* of X^* is compact and Hausdorff.

For a normed space X, we have a natural map $\Phi : X \to X^{**}$ by sending an $x \in X$ to the evaluation at x. Using Hahn-Banach, one shows that it is a linear isometric embedding.

Theorem A.21. Let X be a normed space. And B^* be as in the Banach-Alaoglu theorem. Then Φ composed with the restriction to B^* is a linear isometric embedding into the normed space of continuous functions $C(B^*, \mathbb{K})$ equipped with the supremum norm. In particular, any normed space can be embedded in a $(C(K, \mathbb{K}), \|\cdot\|_\infty)$ for a compact Hausdorff space K.

Proposition A.22. Let A be a $\mathbb{K}$-algebra (which we will always assume associative and unitary). An algebra-morphism $A \to \mathbb{K}$ (we assume it sends 1 to 1) is automatically continuous and has norm 1.

Definition A.23. For a $\mathbb{K}$-algebra A, we denote $\widehat{A}$ for the set of algebra-homomorphisms $A \to \mathbb{K}$. The *Gelfand-transform* of A is the map Φ composed with restriction to $\widehat{A}$: it sends $a \in A$ to the evaluation homomorphism $\widehat{A} \to \mathbb{K}$ at a.

If A is a commutative Banach algebra over $\mathbb{C}$, then $\widehat{A}$ is also called the *maximal ideal space*, because algebra homomorphisms $A \to \mathbb{C}$ correspond bijectively to maximal ideals of A, by taking their kernel.

Proposition A.24. If A is a Banachalgebra, then $\widehat{A} \subseteq B^* \subseteq A^*$ is compact Hausdorff for the induced topology by the product topology. The Gelfand transform $A \to C(\widehat{A}, \mathbb{K})$ is a continuous algebra-morphism with norm at most 1. If A is commutative and $\mathbb{K} = \mathbb{C}$, its kernel is the Jacobson radical of A (the intersection of maximal ideals).

A commutative C^*-algebra is always semi-simple (meaning that the Jacobson radical is zero). More generally, a nonzero normal element of a C^*-algebra is not contained in the Jacobson radical. One way to show this is by introducing the *spectral radius* of an element of a Banach algebra, the supremum of the absolute values of elements of its spectrum. One shows that the spectral radius does not depend on the ambient Banach algebra, and that for normal elements of a C^*-algebra it equals the norm. Using Stone-Weierstrass, one shows:

Theorem A.25. Let A be a commutative C^*-algebra. Then the Gelfand transform is a metric *-isomorphism ($*$ meaning that it commutes with the involution) $A \to C(\widehat{A}, \mathbb{C})$.

A.4 Continuous functional calculus

Theorem A.26 (Functional calculus for normal elements)**.**

1. Let A be a commutative C^*-algebra and $x \in A$ with spectrum $\operatorname{spec} x$. Then there exists a unique metric *-embedding $C(\operatorname{spec} x, \mathbb{C}) \to A$ that sends the inclusion to x; the image of f is denoted $f(x)$.
2. Let A be a C^*-algebra and $x \in A$ normal. Then the same holds.

Sketch of proof.

1. By (A.25) there is an isomorphism $\Gamma : A \to C(\widehat{A}, \mathbb{C})$, which sends x to $\widehat{x}$, the evaluation at x. We have a continuous *-algebra morphism
$$C(\operatorname{spec} x, \mathbb{C}) \to C(\widehat{A}, \mathbb{C})$$
by sending f to $f \circ \hat{x}$. (We use here that the image of $\widehat{x}$ is contained in $\operatorname{spec} x$.) It is an isometry (for the sup-norms) because $\widehat{x} : \widehat{A} \to \operatorname{spec} x$ is surjective. We then apply the *-isomorphism Γ to obtain a metric *-embedding $C(\operatorname{spec} x, \mathbb{C}) \to A$. Uniqueness follows from continuity, because every element of $C(\operatorname{spec} x, \mathbb{C})$ is a (uniform) limit of polynomials in the inclusion and its complex conjugate, by Stone-Weierstrass.
2. By applying the first statement to the commutative C^*-subalgebra generated by x. (The spectrum of an element does not depend on the ambient Banach algebra.) □

This applies in particular to normal operators on a complex Hilbert space. (In which case the bounded operators form a C^*-algebra.)

A.5 Positive operators

Definition A.27 (Positive operator)**.** A bounded operator A on a complex Hilbert space H is *positive* or *non-negative* if the following equivalent conditions hold:

- $\langle Ax, x\rangle \in \mathbb{R}_{\geqslant 0}$ for all $x \in H$.
- A is normal and $\operatorname{spec} A \subseteq \mathbb{R}_{\geqslant 0}$.
- A is of the form B^*B.
- A is of the form BB^*.
- A is the square of a positive self-adjoint operator.

This defines a partial order on bounded operators. Positive implies self-adjoint. The last four conditions are equivalent in any C^*-algebra. On a finite-dimensional vector space, a positive operator is precisely a positive semidefinite operator.

Example A.28. Let K be a compact topological space and consider the C^*-algebra of continuous functions $C(K, \mathbb{C})$. The invertible elements are those functions with values in $\mathbb{C}^\times$. The positive elements are the functions with values in $\mathbb{R}_{\geqslant 0}$.

Functional calculus for normal operators provides a canonical choice for the positive square root (one direction in the above equivalence). It is unique, because a commutative C^*-algebra is isomorphic (via the Gelfand transform) to the algebra of continuous functions on a compact space (its maximal ideal space). We can denote the unique positive square root by $\sqrt{A}$. We also denote

$$|A| = \sqrt{A^*A}$$

A.6 Compact operators

Definition A.29 (Compact operator)**.** Let $T : X \to Y$ be a linear map between normed spaces. Then T is *compact* if the following equivalent conditions hold:

- The image of the unit ball of X is relatively compact.
- The image of every bounded set is relatively compact.
- The image of every bounded sequence contains a convergent subsequence.

And if Y is Banach:

- The image of a bounded set is totally bounded.

A relatively compact subset is bounded, so a compact operator is automatically bounded. A bounded operator followed by a compact operator is by definition compact. A compact operator followed by a bounded one is compact as well: we need that the continuous image of a relatively compact set U under a bounded T is relatively compact. Indeed, we always have $T(\overline{U}) \subseteq \overline{T(U)}$, the reverse inclusion holds because $\overline{U}$ is compact, so its image is, and so it is a closed set containing $T(U)$.

Definition A.30. A *finite-rank operator* is one whose image has finite dimension.

Example A.31. A bounded finite-rank operator is compact, because a bounded subset of a finite-dimensional vector space is relatively compact. The identity map on an inner product space is compact iff the space is finite-dimensional. This holds more generally for normed spaces (Riesz).

The compact operators on a Hilbert space are complete for the operator norm. See [Bump, 1996, Lemma 2.3.1].

Proposition A.32. For a bounded operator T between Hilbert spaces, TFAE:

- T is compact.
- T^* is compact.
- T is the limit of a sequence of finite-rank operators (for the operator norm).

Proof. See [Conway, 1990, Theorem II.4.4]. □

Example A.33 (Compactness of integral operators)**.** Let X be a measure space and $k \in L^2(X \times X)$. Then

$$K : f \mapsto \int k(\cdot, y) f(y) dy$$

is compact on $L^2(X)$ with norm at most $\|k\|_2$.

Sketch of proof. We know that it is bounded by (A.7). For compactness, we can construct a sequence of finite rank operators that converges to K. See [Conway, 1990, Proposition II.4.7]. □

Such an integral operator is even Hilbert–Schmidt (A.56).

Theorem A.34 (Fredholm alternative)**.** Let T be a compact operator on a Hilbert space and $\lambda \in \mathbb{C}^\times$. TFAE:

1. $T - \lambda$ is injective: λ is not an eigenvalue of T.
2. $T - \lambda$ is surjective.
3. $T - \lambda$ is invertible.

Proof. This is part of the spectral theory of compact operators; see e.g. [Conway, 1990, pp. VII.7.9, IVI.7.10]. On finite-dimensional spaces this is the rank-nullity theorem. □

Proposition A.35 (Eigenvalues of a compact operator)**.** Let A be a compact operator on a $\mathbb{K}$-Hilbert space H. The eigenspaces corresponding to nonzero eigenvalues have finite dimension, there are only countably many nonzero eigenvalues and they tend to 0.

Proof. [Conway, 1990, Proposition II.4.13] We prove all statements at once. The image of any infinite orthonormal set of eigenvectors is relatively compact and thus contains a Cauchy sequence. If there were uncountably many eigenvalues $\lambda_i \neq 0$, then there would exist $n \in \mathbb{N}$ with infinitely many $|\lambda_i| > 1/n$. If countably many are nonzero but they do not tend to 0, the same holds. In both cases, and also if an eigenspace for some $\lambda_i \neq 0$ is infinite-dimensional, there is an infinite orthonormal set of eigenvectors e_i corresponding to $|\lambda_i| > \delta > 0$. But

$$\|\lambda_i e_i - \lambda_j e_j\| = |\lambda_i|^2 + |\lambda_j|^2 > \delta^2 \tag{A.36}$$

meaning that the image of the infinite bounded set of (e_i), contains no Cauchy sequence. □

The same holds true more generally in a Banach space, and one can show:

Theorem A.37 (Spectral theory of compact operators)**.** Let X be a $\mathbb{K}$-Banach space and T a compact operator on X.

1. Every nonzero element of its spectrum $\lambda \in \sigma(T)$ is an eigenvalue of T.
2. The subspaces corresponding to nonzero eigenvalues are finite-dimensional.
3. The spectrum of T is countable and can only have 0 as an accumulation point.

Sketch of proof. The essential ingredient is (the elementary) Riesz' lemma: Given a non-dense proper subspace $Y \subset X$ and $\epsilon > 0$, there is $x \in X$ of norm 1 and at distance at least $1 - \epsilon$ from Y. One can think of this as being a substitute for the inequality (A.36). One then exploits this together with the compactness of T; the proof is elementary. See e.g. [Conway, 1990, VII§7]. □

A.7 Diagonalizable operators

Definition A.38. A *projection* on a $\mathbb{K}$-Hilbert space is a self-adjoint idempotent bounded operator. Two projections P, Q are *orthogonal* if $PQ = 0$, equivalently, if $QP = 0$.

Proposition A.39 (Properties of projections)**.** We work on a $\mathbb{K}$-Hilbert space H.

1. For a projection P we have $\ker P = \operatorname{ran}(1 - P)$ and $\ker(1 - P) = \operatorname{ran}(P)$; they are closed subspaces.

2. Projections are in 1-1-correspondence with closed linear subspaces, by sending P to its range, and a closed subspace V to the orthogonal projection on it, which comes from the decomposition $H = V \oplus V^\perp$.

3. Under this bijection, orthogonal projections correspond to orthogonal subspaces.

4. If P is the projection on V, then $1 - P$ is the projection on its orthogonal complement.

5. Let (P_i) be a family of pairwise orthogonal projections on (V_i) and P be the projection on their span V. Then for all $x \in H$,
$$Px = \sum_i P_i x$$
where the RHS is a series with countable support which converges absolutely.

Definition A.40 (Diagonalizable operator)**.** Let H be a Hilbert space and A a bounded operator on H. Then A is *diagonalizable* if the following equivalent conditions hold:

1. There exist closed pairwise orthogonal eigenspaces for A that span H, i.e. H is the orthogonal direct sum of eigenspaces.

2. H has an orthogonal basis of eigenvectors for A.

3. There exist pairwise orthogonal projections (P_i) and $\lambda_i \in \mathbb{C}$ such that for all $x \in H$,
$$Ax = \sum_i \lambda_i P_i x$$
where the RHS is a series with countable support which converges absolutely.

4. There exist pairwise orthogonal projections (P_i) and *pairwise distinct* $\lambda_i \in \mathbb{C}$ such that for all $x \in H$,
$$Ax = \sum_i \lambda_i P_i x$$
where the RHS is a series with countable support.

A family of operators is *simultaneously diagonalizable* if the choice in each or any of these definitions can be made independently of the operator.

Remark A.41.

1. This does not imply $A = \sum_i \lambda_i P_i$. Indeed, the P_i have norm 1 so this sum can only converge if $\lambda_n \to 0$. This is also sufficient; see (A.43).

2. This definition is stronger than the usual notion of diagonalizability of linear operators on $\mathbb{C}^n$ or $\mathbb{R}^n$.

Proposition A.42 (Properties of diagonalizable operators)**.** Let A be a bounded diagonalizable operator on a Hilbert space, then:

1. It is normal.

2. Its eigenvalues λ_i are bounded, and $\|A\| = \sup |\lambda_i|$.

3. It is compact iff only countably many eigenvalues (with multiplicites) are nonzero, and they go to 0 (counting multiplicities). In particular, eigenspaces for nonzero eigenvalues are finite-dimensional.

4. Positive iff its eigenvalues are real and $\geqslant 0$.

Sketch of proof. Let (e_i) be an orthonormal basis in which A is diagonal.

1. Because its adjoint is diagonalizable in the same basis (with the conjugate eigenvalues).

2. Take an element $\sum \mu_i e_i$, then its image has norm at most $\sup |\lambda_i|$ times the original norm. The bound is tight by taking eigenvectors.

3. [Conway, 1990, Proposition II.4.6] If A is compact, this follows from the more general (A.35). Conversely, when they go to zero, we can write the operator as a limit of finite rank operators and conclude by (A.32). Indeed, order the countably many nonzero eigenvalues (λ_i) with multiplicities, and corresponding eigenvectors (e_i). If P_i denotes the orthogonal projection on $\langle e_i \rangle$, then
$$\left\| A - \sum_{i=1}^{n} \lambda_i P_i \right\| = \sup_{i>n} |\lambda_i|$$
which goes to 0 when $n \to \infty$.

4. Because if $x = \sum \mu_i e_i$ then $\langle Ax, x \rangle = \sum |\mu_i|^2 \lambda_i$. □

To come back to (A.41), we have:

Proposition A.43. Let H be a Hilbert space, A a diagonalizable operator with countably many eigenvalues λ_i so that there exist pairwise orthogonal projections (P_i) such that for every $x \in H$:
$$Ax = \sum_i \lambda_i P_i x$$
Then $\lambda_n \to 0$ iff $A = \sum \lambda_i P_i$.

In particular, this is the case if A is compact, and it implies that A is compact *if* in addition the eigenspaces for nonzero eigenvalues are finite-dimensional: then $\lambda_n \to 0$ with multiplicities, and A is compact by (A.42).

Proof. For a finite family of pairwise orthogonal projections (P_i) and $\lambda_i \in \mathbb{C}$ we have
$$\left\| \sum \lambda_i P_i \right\| = \max |\lambda_i|$$
Thus using Cauchy's criterion, $\lambda_n \to 0$ implies that $\sum \lambda_i P_i$ converges, and by continuity of the evaluation $(A, x) \mapsto Ax$, it equals A. Conversely, $\|\lambda_i P_i\| = |\lambda_i|$, so for the series to converge we need $\lambda_n \to 0$. □

Theorem A.44 (Simultaneous diagonalizability)**.** Let H be a Hilbert space.

1. The restriction of a diagonalizable (bounded) operator to an invariant closed subspace is diagonalizable

2. Let $(A_i)_{i \in I}$ be a family of commuting diagonalizable (bounded) operators. Then they are simultaneaously diagonalizable.

Proof. 1. Because the restriction of a projection is a projection.

2. Commuting operators stabilize each other's eigenspaces, and thus intersections thereof. If the family is finite, we can conclude by induction: diagonalize all A_i except A_1, and then the restriction of A_1 to the intersections of their eigenspaces. For general families, the same argument works: we use Zorn's lemma and show that a maximal subset $J \subseteq I$ for which the $(A_j)_{j \in J}$ are simultaneously diagonalizable, is necessarily the whole of I. □

Proposition A.45 (Properties of simultaneously diagonalizable operators)**.** Let H be a $\mathbb{C}$-Hilbert space. The sum and product of simultaneously diagonalizable operators A, B is diagonalizable in the same basis.

Proof. By assumption, H has an orthonormal basis of eigenvectors for both A and B, which are visibly eigenvectors for $A + B$ and AB. □

A.8 Spectral theory of compact normal operators

Proposition A.46 (Isolated points of the spectrum)**.** Let H be a Hilbert space and A a normal bounded operator. Let λ be an isolated point of its spectrum so that the characteristic function 1_λ is continuous on $\operatorname{spec}(A)$. Then

1. The element $P = 1_\lambda(A)$ given by the functional calculus of (A.26) is the orthogonal projection on the eigenspace $\ker(A - \lambda)$.

2. λ is an eigenvalue of A.

Proof. [Vernaeve, 2015, Stelling 2.7.11; Kowalski, 2009, Corollary 3.8] We have that 1_λ is self-adjoint and idempotent, hence so is P: it is an orthogonal projection.

1. We want that $Ax = \lambda x$ iff $Px = x$, for $x \in H$. By functional calculus it suffices that $\operatorname{id} - \lambda$ and $1 - 1_\lambda$ divide each other in $C(\operatorname{spec}(A), \mathbb{C})$. Indeed: if f divides g in $\operatorname{spec} A$ then $\ker f(A) \subseteq \ker g(A)$. Note that $f = (1 - 1_\lambda)/(\operatorname{id} - \lambda)$ defines a continuous nonzero function on $\operatorname{spec} A - \{\lambda\}$, and we can extend f and its reciprocal continuously at λ by assigning any value to the image of λ. Hence $\operatorname{id} - \lambda$ and $1 - 1_\lambda$ divide each other.

2. The eigenspace for λ is the range of P. Because $1_\lambda \neq 0$, we have $P \neq 0$, so the range is nonzero. □

Theorem A.47 (Spectral theorem for normal operators with almost discrete spectrum)**.** Let H be a complex Hilbert space and A a normal operator on H. Suppose that the spectrum $\operatorname{spec}(A)$ is discrete or has only 0 as an accumulation point, (so that it is in particular countable). Then A is diagonalizable, and if the P_i are the projections on the eigenspaces with eigenvalues λ_i, then

$$A = \sum \lambda_i P_i$$

Proof. [Vernaeve, 2015, Stelling 2.7.11] Because $\operatorname{spec}(A)$ is compact, we have $\lambda_n \to 0$, at least if there are infinitely many eigenvalues. We treat the finite case in the same breath, all statements about convergence being trivial in that case. For nonzero $\lambda_n \neq 0$, we have $1_{\lambda_n}(A) = P_n$ by (A.46) and the functions 1_{λ_n} and 1_{λ_m} are orthogonal for $n \neq m$, so that the projections P_n and P_m are orthogonal. Because $\lambda_n \to 0$, we have $1 = \sum \lambda_n 1_{\lambda_n}$ in $C(\operatorname{spec} A, \mathbb{C})$ and thus

$$A = \sum \lambda_n P_n$$

by (A.26). □

By (A.37), the above holds in particular for compact normal operators, and thus for compact self-adjoint operators:

Theorem A.48 (Spectral theorem for compact normal operators)**.** A compact normal operator on a Hilbert space is diagonalizable, has countably many eigenvalues, and finite-dimensional eigenspaces for nonzero eigenvalues.

Proof. The only new information is that eigenspaces for nonzero eigenvalues have finite dimension. This follows from (A.35). □

Theorem A.49 (Spectral theorem for compact self-adjoint operators)**.** A compact self-adjoint operator on a Hilbert space is diagonalizable, has countably many eigenvalues, and finite-dimensional eigenspaces for nonzero eigenvalues.

Alternative proof of (A.49). One can prove diagonalizability by successively exhibiting eigenvectors, and reducing to a compact operator with smaller norm. The key argument is that a compact self-adjoint operator has an eigenvalue whose absolute value equals the spectral radius. See e.g. [Conway, 1990, Theorem II.5.1] or [Bump, 1996, Theorem 2.3.1]. □

Proof of (A.48) from (A.49). [Conway, 1990, Theorem II.7.6] Because A is normal,

$$X = \Re\mathrm{e}\, A := \frac{A + A^*}{2} \quad \text{and}$$
$$Y = \Im\mathrm{m}\, A := \frac{A - A^*}{2i}$$

which are always self-adjoint, commute. By (A.44), X and Y are simultaneously diagonalizable, hence $A = X + iY$ is diagonalizable. □

As a corollary, the square root of a positive compact operator is again compact. In particular, the absolute value $|A|$ of a compact operator A is compact.

A.9 Trace class and Hilbert–Schmidt operators

The below can be found in [Conway, 1990, Exercise IX.2.19–20].

Definition A.50 (Trace of a positive operator)**.** Let A be a positive operator on a complex Hilbert space with orthonormal basis (e_i). Its *trace* is the sum of nonnegative terms

$$\operatorname{Tr} A := \sum_i \langle Ae_i, e_i \rangle \in [0, +\infty]$$

It does not depend on the choice of the basis.

Definition A.51. A *Hilbert–Schmidt operator* A between Hilbert spaces is a bounded operator for which the positive operator A^*A has finite trace. That is, the sum

$$\|A\|_2^2 := \sum_i \|Ae_i\|^2 = \sum_{i,j} |\langle Ae_i, f_j \rangle|^2 < \infty$$

is finite in some (or any) orthonormal bases (e_i) and (f_j), and $\|A\|_2$ is called the *Hilbert–Schmidt norm*.

Proposition A.52 (Properties of Hilbert–Schmidt operators)**.** On a Hilbert space, we have that:

1. Hilbert–Schmidt operators form a two-sided ideal of the algebra of bounded operators, which is stable by taking adjoints.
2. Finite rank implies Hilbert–Schmidt implies compact. The finite rank operators are dense in the Hilbert–Schmidt.
3. $\|\cdot\|_2$ is a norm and and $\|A^*\|_2 = \|A\|_2 \geqslant \|A\|$
4. If A, B are Hilbert–Schmidt and AB positive, then AB has finite trace. If in addition BA is positive, then $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.
5. A diagonalizable operator with eigenvalues λ_i is Hilbert–Schmidt iff $\sum |\lambda_i|^2 < \infty$ (counting multiplicities).

Definition A.53. A *trace class operator* A on a Hilbert space is one for which the following equivalent conditions hold:

1. $|A|$ is the product of Hilbert–Schmidt operators.
2. $|A|$ has finite trace.

3. $\sqrt{|A|}$ is Hilbert–Schmidt.

4. A is the product of Hilbert–Schmidt operators.

Proposition A.54 (Properties of trace class operators)**.** On a Hilbert space with orthonormal basis (e_i), we have that:

1. Trace class operators form a two-sided ideal of the algebra of bounded operators, which is stable by taking adjoints.

2. Finite rank implies trace class implies Hilbert–Schmidt implies compact. The finite rank operators are dense in the trace class.

3. If A is trace class, then the trace
$$\operatorname{Tr} A := \sum_i \langle Ae_i, e_i\rangle$$
converges (in particular, only countably many terms are nonzero) and is independent of the basis.

4. $\operatorname{Tr}(B^*A)$ defines an inner product on Hilbert–Schmidt operators whose norm is $\|\cdot\|_2$. The Hilbert–Schmidt form a Hilbert space for this inner product.

5. Trace class is a Banach space for the norm $\|\cdot\|_1$ defined by $\|A\|_1 = \operatorname{Tr}|A|$.

6. A diagonalizable operator with eigenvalues λ_i is trace class iff $\sum|\lambda_i| < \infty$ (counting multiplicities).

Remark A.55 (The trace for non-trace-class operators)**.** If A is an operator on a Hilbert space H with orthonormal bases (e_i) and (f_j), then the absolute convergence of $\sum\langle Ae_i, e_i\rangle$ does not imply that of $\sum\langle Af_j, f_j\rangle$: take H separable and A diagonal in the basis $(f_j)_{j\geqslant 1}$ with eigenvalues $(-1)^j/j$. Let $e_{2i+1} = (f_{2i+1} + f_{2i+2})/\sqrt{2}$ and $e_{2i+2} = (f_{2i+1} - f_{2i+2})/\sqrt{2}$. Then
$$\sum\langle Ae_i, e_i\rangle = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2n+2} - \frac{1}{2n+1}\right)$$
converges absolutely, but
$$\sum_j\langle Af_j, f_j\rangle = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$
does not. Moreover, rearranging the f_j can make the sum converge to anything, by Riemann's rearrangement theorem. In short, the trace does not make sense for non-trace-class operators.

Example A.56 (Hilbert–Schmidt integral operators)**.** Let X be a measure space and $k \in L^2(X \times X)$. Then
$$K : f \mapsto \int k(\cdot, y) f(y) dy$$
is Hilbert–Schmidt on $L^2(X)$ with Hilbert–Schmidt norm $\|K\|_2$ equal to the L^2 norm $\|k\|_2$.

Proof. [Conway, 1990, Lemma II.4.8; Bump, 1996, Theorem 2.3.2]. For $\|K\|_2 = \|k\|_2$ we use the fact that if (e_i) is an orthonormal basis of $L^2(X)$, then the $e_i(x)e_j(y)$ are an orthonormal basis of $L^2(X \times X)$. □

Conversely, one can show that if L^2 is separable, every Hilbert–Schmidt operator is of the above form. It is self-adjoint iff $k(x, y) = \overline{k(y, x)}$ a.e.

B Functional calculus

We are often confronted with complex-valued functions on $M \times U$, where M is a smooth manifold and $U \subseteq \mathbb{C}$ is an open set, and we are interested in their holomorphic or meromorphic dependence on the second variable. There are many different ways to interpret "holomorphic dependence". For example:

Definition B.1. We call $f : M \times U \to \mathbb{C}$ *pointwise holomorphic (meromorphic)*[12] if $f(w, \cdot)$ is holomorphic (meromorphic) for all $w \in M$. We call f *uniformly meromorphic*[12] if there exists a closed discrete set $P \subset U$ and a map $m : P \to \mathbb{N}_{>0}$ such that each $f(w, \cdot)$ is meromorphic with poles contained in P, and with the order of $p \in P$ at most $m(p)$.

Using Weierstrass products, one can construct a holomorphic function on U (not identically zero on each connected component), which has a zero of multiplicity $m(p)$ at each $p \in P$. Thus uniform meromorphy of $f(w, s)$ is equivalent to the existence of a holomorphic g (not a zero divisor) such that $f(w, s)g(s)$ is pointwise holomorphic.
When $f(w, s)$ is pointwise holomorphic and smooth for fixed s, it is reasonable to require that when D is a differential operator on M, then $Df(w, s)$ is still pointwise holomorphic. Because holomorphic functions are nothing else than differentiable functions of two real variables annihilated by the Cauchy–Riemann operator

$$\frac{\partial}{\partial \bar{s}} = \frac{1}{2}\left(\frac{\partial}{\partial \sigma} + i\frac{\partial}{\partial t}\right)$$

we see that Df will still be pointwise holomorphic when f is jointly smooth, seen as a function on $M \times \mathbb{R}^2$:

Definition B.2. Given topological spaces X, Y, Z and a map $f : X \times Y \to Z$, we call it *separately continuous* if for all $x_0 \in X$ and $y_0 \in Y$ the maps

$$f(x_0, \cdot) : Y \to Z$$
$$f(\cdot, y_0) : X \to Z$$

are continuous, and *(jointly) continuous* if $f : X \times Y \to Z$ is continuous. When X, Y, Z are smooth manifolds, we define *separate smoothness* and *(joint) smoothness* similarly.

Similarly, joint smoothness implies that the complex derivatives $f^{(n)}(w, s)$ are still smooth for fixed s. It turns out that we don't need joint smoothness for that:

Proposition B.3 (Joint regularity of complex derivatives)**.** Let M be a smooth manifold and $U \subseteq \mathbb{C}$ open. Let $f : M \times U \to \mathbb{C}$ be pointwise holomorphic. Then:

1. If f is jointly (resp. separately) continuous, then so is f'.
2. If f is of class C^1 then so is f', in which case for every chart (x^i) of M we have that $\partial f/\partial x^i$ is pointwise holomorphic and
$$\frac{\partial}{\partial s}\frac{\partial}{\partial x^i} f = \frac{\partial}{\partial x^i}\frac{\partial}{\partial s} f$$
3. If for every chart (x^i) the partial derivatives $\frac{\partial^{|\alpha|}}{\partial x^\alpha} f(w, s)$ up to order n exist and are continuous, then f is jointly of class C^n.

Proof. 1. Suppose f is (jointly) continuous. Fix $(w_0, s_0) \in M \times U$. We have, for s in a small compact neighborhood V of s_0:

$$f'(w, s) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(w, \zeta)}{(\zeta - s)^2} d\zeta$$

[12] The terminology is new, we are introducing it purely for convenience.

where B is some small ball centered at s_0. If we restrict w to a compact neighborhood W of w_0, then $f(w,\zeta)/(\zeta - s)^2$ becomes uniformly continuous for $(w, s, \zeta) \in W \times V \times \partial B$, so that the above integral defines a continuous function of (w, s).

Similarly for separate continuity, or by noting that it follows from the case of joint continuity by taking $M = \{w\}$ of dimension zero.

2. Because f is C^1, the integral of $\partial f/\partial x^i(w, \cdot)$ along closed contractible contours is still 0, hence it is holomorphic for all w. The equality follows from Cauchy's integral formula, from which we also see that f' is still C^1.

3. We are in particular assuming that $f = \partial^0 f/\partial x^0$ is continuous. We proceed by induction on n. For $n = 1$, the partial derivatives of order 1 are indeed continuous by 1. and 2. If $n > 1$, we note that

   (i) each $\partial f/\partial x^i$ is pointwise holomorphic and has continuous partial derivatives w.r.t. w up to order $n - 1$

   (ii) f' is pointwise holomorphic, and using Cauchy's integral formula one shows that its partial derivatives up to order n exist and are continuous.

   and conclude by induction. □

Compare this with Hartog's lemma, which says that a separately holomorphic function is jointly holomorphic (without any assumption of joint continuity). Such 'separate versus joint' properties have been extensively studied. We will not get any deeper into that. We only remark that a separately continuous function $\mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$ is Lebesgue-measurable [Johnson, 1969].
When studying the spectrum of a bounded operator on a complex Banach space, or more generally, of an element of a C^*-algebra, we are also confronted with functions that take values in a Banach algebra. More generally, one can wonder about holomorphic functions with values in a topological complex vector space. Turning back to the first question, given a smooth function $f : M \times U \to \mathbb{C}$, we could then take 'holomorphic dependence' to mean that $f : U \to C^\infty(M)$ is holomorphic. These notions and their properties are the subject of this section.

B.1 Differentiability and holomorphy

Let $\mathbb{K}$ be any of the fields $\mathbb{R}$ or $\mathbb{C}$. All our topological vector spaces will be assumed Hausdorff.

Definition B.4. Let X, Y be topological $\mathbb{K}$-vector spaces, $U \subseteq X$ open and $f : U \to Y$ a function.

1. Let X, Y be Banach spaces. We call f (Fréchet) differentiable at $x \in U$ if it is approximately linear at x:
$$f(x + h) = f(x) + Ah + o(h) \quad (h \to 0)$$
   for some bounded linear $A : X \to Y$, the *(Fréchet) differential*.

2. We will be almost exclusively interested in the case where $X = \mathbb{K}$, in which case the definition generalizes to topological $\mathbb{K}$-vector spaces Y: we call f differentiable at $x \in U$ if the limit
$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
   exists, in which case we call it the derivative. If $\mathbb{K} = \mathbb{C}$ and f is complex differentiable in an open set $U \subseteq X$, we also call it *(strongly) holomorphic* in U.

3. Let X and Y be general topological $\mathbb{K}$-vector spaces. Suppose $x = 0 \in U$ and $f(0) = 0$. We call f *tangent at* 0 if for every neighborhood Ω of $0 \in Y$ there exists a neighborhood V of $0 \in X$ such that $V \subseteq U$, and a neighborhood I of $0 \in \mathbb{K}$ such that

$$f(tV) \in o(t)\Omega \qquad (t \in I) \tag{B.5}$$

for some $o(t)$-function $I \to \mathbb{K}$, depending on Ω and V. For general x and $f(x)$, we call f differentiable at x if there exists a continuous linear map $A : X \to Y$ such that $h \mapsto f(x+h) - f(x) - Ah$ is tangent at 0.

Proof of equivalence. It is not immediately clear why the third definition generalizes the others.

$3 \implies 2$: Take a neighborhood Ω of $0 \in Y$, so that $f(x) + \Omega$ is a neighborhood of $f(x)$. Let A, V and I be as in the hypothesis of definition 3. Then $f(x+th) - f(x) - Ath \in o(t)\Omega$ for $h \in V$, $t \in I$. Fix one such $h \neq 0$, and take $t = k/h$, with $k \in \mathbb{K}$ sufficiently small so that $k/h \in I$. Then

$$\frac{f(x+k) - f(x)}{k} - A \in \frac{o(k/h)}{k}\Omega \qquad (k \to 0)$$

Now note that if Y is not locally convex, there is no reason to assume that, for example, $B(0,1) \cdot \Omega \subseteq \Omega$.

Take any neighborhood Ω' of $0 \in Y$. By continuity of scalar multiplication, there exists a neighborhood Ω of $0 \in Y$ and a neighborhood J of 0 in $\mathbb{K}$ such that $J\Omega \subseteq \Omega'$. Then $\frac{o(k/h)}{k}\Omega \subseteq \Omega'$ for k sufficiently small, with h and $o(k/h)$ as above. We conclude that $\frac{f(x+k)-f(x)}{k} - A \in \Omega'$ for small k. Because Ω' was arbitrary,

$$\lim_{k \to 0} \frac{f(x+k) - f(x)}{k} = A$$

$2 \implies 3$: Take a neighborhood Ω of $0 \in Y$. By assumption, there exists $A \in Y$ (independent of Ω) with $f(x+h) - f(x) - Ah \in h\Omega$ for h sufficiently small, say $h \in B(0,\delta) =: I$. Let $V = B(0,1)$. For h even smaller, we have $f(x+h) - f(x) - Ah \in h\Omega/2$. (But we are not using that $\Omega/2 \subseteq \Omega$; this need not be the case). And so on: we find a decreasing sequence $\delta_n \in \mathbb{R}_{>0}$ with $\delta_1 = \delta$ such that $f(x+h) - f(x) - Ah \in h\Omega/n$ for $h \in B(0,\delta_n) = \delta_n V$. Hence we can define $o(t) : I \to \mathbb{K}$ to be t/n on the annulus $B(0,\delta_n) - B(0,\delta_{n+1})$, and 0 elsewhere, should $\delta_n \not\to 0$.

$3 \Leftrightarrow 1$: This is immediate, using the fact that balls form a basis of the topology. □

We will rarely work with the general notion for topological vector spaces. When X and Y are $\mathbb{K}$-Banach spaces, the set of bounded linear maps $L(X,Y)$ is again a Banach space, so if f is $\mathbb{K}$-differentiable in U it makes sense to ask about the second order derivative, and so on. The usual facts carry trough, with the same proofs: differentiable at x implies continuous at x, linear maps are everywhere differentiable and equal to their differential at every point, and we have a chain rule. If $X = \mathbb{K}$ and Y is a topological $\mathbb{K}$-vector space, we have a canonical identification $L(X,Y) \cong Y$ and we can talk about (higher) derivatives as elements of Y. We also have a product rule and a quotient rule.

Proposition B.6. Let X be a $\mathbb{K}$-Banach space, Y a $\mathbb{K}$-Banach algebra, $U \subseteq X$ open and let $f, g : U \to Y$ be $\mathbb{K}$-differentiable at $x_0 \in U$.

1. Then fg is $\mathbb{K}$-differentiable at x_0 and we have the product rule

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

2. Suppose $f(x_0)$ is invertible. Then $f(x)$ is invertible for x in a neighborhood of x_0, and the inverse $1/f$ is $\mathbb{K}$-differentiable at x_0 with derivative

$$(1/f)'(x_0) = -f(x_0)^{-1} f'(x_0) f(x_0)^{-1}$$

Proof. 1. The proof is the same as for functions $\mathbb{C} \to \mathbb{C}$.

2. By the chain rule, it suffices to prove that $y \mapsto y^{-1}$ is $\mathbb{K}$-differentiable at invertible elements of Y, with derivative $h \mapsto -y_0^{-1} h y_0^{-1}$. Note that indeed, if y_0 is invertible, then elements in a

neighborhood of y_0 are invertible (when they are at distance less than $\left\|y_0^{-1}\right\|^{-1}$) so that y^{-1} is defined on a neighborhood of y_0. For such y we have

$$\begin{aligned}
y^{-1} - y_0^{-1} &= \left(y_0(1 - y_0^{-1}(y_0 - y))\right)^{-1} - y_0^{-1} \\
&= (1 - y_0^{-1}(y_0 - y))^{-1} y_0^{-1} - y_0^{-1} \\
&= \left(1 + y_0^{-1}(y_0 - y) + o(y_0 - y)\right) y_0^{-1} - y_0^{-1} \\
&= y_0^{-1}(y_0 - y) y_0^{-1} + o(y_0 - y)
\end{aligned}$$

because we can Taylor expand $(1-x)^{-1}$ as soon as $\|x\| < 1$. The conclusion follows. □

The product rule is, visibly, more generally valid when Y is a topological $\mathbb{K}$-algebra. We are not attempting to state the results in the most general possible settings.

B.2 Weak holomorphy

Everything we said about differentiability so far was elementary. The theory of holomorphic functions in topological vector spaces is rich. Note that a function $\mathbb{C} \to \mathbb{C}^m$ is holomorphic iff all of its components are. One can generalize this and define:

Definition B.7. Let X be a complex topological vector space, $U \subseteq \mathbb{C}$ open and $f = U \to X$ a function. Then f is *weakly holomorphic* in U if for all $\phi \in X^*$, the $\mathbb{C}$-valued function $\phi \circ f : U \to \mathbb{C}$ is holomorphic.

It turns out that the converse holds:

Theorem B.8. Let $U \subseteq \mathbb{C}$ open.

1. (Dunford) If X is a Banach space, a weakly holomorphic $f : U \to X$ is strongly holomorphic.

2. More generally, this holds if X is a topological vector space, locally convex (its topology is induced by a set of seminorms) and quasi-complete (meaning that every closed bounded subset is complete).

Proof. 1. See [Yosida, 1980, §V.3, Theorem 1]. The proof uses a Cauchy integral formula in Banach spaces via Bochner-integration, discussed briefly below.

2. See [Garrett, 2005]. The proof uses the more general notion of Gelfand–Pettis integration. □

This implies in particular to Fréchet spaces (complete locally convex Hausdorff vector spaces whose topology is induced by a countable family of seminorms). A Banach space is Fréchet; its topology is induced by only one (semi)norm. We will almost exclusively work with Fréchet spaces.
Before giving more background about integration in topological vector spaces, we illustrate the power of this equivalence. Many results about holomorphic functions $U \to \mathbb{C}$ generalize immediately to Banach spaces:

Corollary B.9. Let X be a complex Fréchet space, $U \subseteq \mathbb{C}$ open and $f : U \to X$. TFAE:

1. f is complex differentiable (i.e. strongly holomorphic)

2. f has all higher derivatives

Corollary B.10. The locally uniform limit f of holomorphic functions $f_n : U \to X$ is holomorphic ...

Corollary B.11 (Hurwitz's theorem)**.** ... in which case f' is the locally uniform limit of f_n'.

In particular, we can differentiate a uniformly convergent series term-wise.

Proof. The first corollary is immediate. For the second, there's a subtlety. Suppose X is a Banach space, for convenience. We will sketch a proof for more general X later. Let $K \subseteq U$ be compact. For $\lambda \in X^*$ we have $f_n \to f$ uniformly on K, so $\lambda \circ f_n \to \lambda \circ f$ uniformly on K, and by Hurwitz, $\lambda \circ f_n' \to \lambda \circ f'$ uniformly on K. Moreover, the convergence is uniform in λ as long as $\|\lambda\|$ remains bounded. Now suppose $\exists \epsilon > 0 : \forall N \in \mathbb{N} : \exists n \geqslant N \exists x_n \in K : \|f_n'(x_n) - f'(x_n)\| > \epsilon$. We use (A.11) to find $\lambda_n \in X^*$ which sends $f_n'(x_n) - f'(x_n)$ to 1 and has norm at most $1/\epsilon$. Then $\|\lambda_n\|$ is bounded, yet the convergence is not uniform in those λ_n. Contradiction. □

Alternatively, one can build a theory of integration, prove a Cauchy formula for f' and directly mimic the proof of Hurwitz's theorem in the general setting. We include the main elements of the approach.

B.3 Three notions of integration

There are different notions of integration in topological vector spaces. We will not use any deep results about them and in fact all we need is some notion of integration, and a way to estimate integrals using a triangle inequality. We present three approaches.
Given a measure space S and a topological $\mathbb{K}$-vector space X, the least we can expect is that integration commutes with continuous linear functionals. That is, for $f : S \to X$:

$$\lambda\left(\int_S f\right) = \int_S \lambda \circ f \qquad \forall \lambda \in X^*$$

We call f *weakly integrable* if such a vector $\int_S f$ exist, in which case we call it a *weak integral* or *Gelfand–Pettis* integral. From the definition, we have: when $T : X \to Y$ is a continuous linear map between topological vector spaces and f has a weak integral $\int_S f$, then $T(\int_S f)$ is a weak integral of $T \circ f$. When X is locally convex, the Hahn–Banach separation theorem implies that continuous linear functionals separate points, hence there can exist at most one such $\int_S f$. One can show that:

Theorem B.12 (Existence of Gelfand–Pettis integrals)**.** Let S be a locally compact Hausdorff topological space with a finite positive Borel measure. Let X be a locally convex and quasi-complete complex topological vector space. Then a compactly supported continuous $f : S \to X$ is weakly integrable.

In particular, we can take S a Lebesgue-measurable subset of some $\mathbb{R}^n$, of finite measure, and we can take X to be a Fréchet space or Banach space. Local convexity should be thought of as requiring that convex linear combinations of small vectors are still small.

Proof. See [Garrett, 2014b, Theorem 1.0.1]. The proof is nonconstructive in that it uses compactness of a certain set, to show that the intersection of a certain family of closed sets is nonempty. This compactness, in its turn, relies on Tychonoff's theorem (and thus on the axiom of choice) via [Garrett, 2014b, Proposition 4.0.1]. □

The above approach differs quite substantially from the theory of Riemann or Lebesgue integration. But they do have infinite dimensional analogues.
The Riemann integral over an interval $[a, b] \subseteq \mathbb{R}$ is defined in the same way as for real-valued functions. Given a topological vector space X and a function $f : [a, b] \to X$ we can define the Riemann sum for every finite partition (subdivision) of $[a, b]$ and for every choice of 'tags', which are points in the closed subintervals defined by the partition. Partitions of $[a, b]$ form a directed set for the relation of being a refinement, and every choice of tags defines a net on the set of partitions, whose values are Riemann sums. We call f Riemann-integrable if the net of Riemann sums converges to a common vector $\int_a^b f \in X$ for every choice of tags. Because the Riemann-sums are linear in f, it follows that a Riemann-integral (when it exists) is a Gelfand–Pettis integral. One can show:

Theorem B.13 (Existence of Riemann-integrals)**.** If X is a complete and locally convex $\mathbb{R}$-vector space, then any continuous $f : [a, b] \to X$ is Riemann-integrable. Moreover, the nets converge in a uniform way, in the following sense. Denote for a partition Δ of $[a, b]$ the maximum distance between two adjacent points of the subdivision by $|\Delta|$. Then for every neighborhood U of 0 in X, there exists $\epsilon > 0$ such that the Riemann sums for all partitions Δ with $|\Delta| < \epsilon$ and all choices of tags, lie in U.

Sketch of proof. For detailed computations, see [Nagy, 2014]. First, one uses local convexity and uniform continuity of f to prove that, for every neighborhood U of 0, there exists $\epsilon > 0$ such that if $|\Delta| < \epsilon$ and Σ is a refinement of Δ, then the difference between their Riemann sums lies in V. (No matter the choice of tags.)
Then take a sequence $\epsilon_n \to 0$ and successive refinements Δ_n with $|\Delta_n| < \epsilon_n$ (and arbitrary tags). Use the first claim to conclude that the sequence of Riemann sums is Cauchy, hence convergent by completeness.
Now take a neighborhood U of 0, take ϵ as in the first claim and Δ with $|\Delta| < \epsilon$. Considering a common refinement Σ_n of Δ and Δ_n, and using the triangle inequality, one shows that the Riemann sums for Δ (for any choice of tags) lie in $2U$, and the conclusion follows. □

If f is continuous, then from the definition it follows that $\int p \circ f \leqslant p(\int f)$ for every continuous seminorm p of X.
In order to generalize the Lebesgue-integral, we need a notion of measurability. Let (S, μ) be a measure space and X a $\mathbb{K}$-Banach space. We call $f : S \to X$ *weakly measurable* if $\lambda \circ f$ is measurable for every $\lambda \in X^*$, and *strongly measurable* if it is a.e. equal to the pointwise limit of a sequence of simple functions. Here, simple functions are defined as for real-valued functions: f is simple if its support has finite measure and if there exists a finite measurable partition into sets on which f is constant. If $I \subseteq \mathbb{R}$ is an interval, a continuous map $I \to X$ is strongly measurable. (The argument is the same as for real-valued functions.)
A strongly measurable function is visibly weakly measurable, and its image must be separable (have a countable dense subset). In fact:

Theorem B.14 (Pettis)**.** A function $f : S \to X$ is strongly measurable iff it is weakly measurable and there exists a subset $T \subseteq S$ whose complement has measure 0, and such that the image $f(T)$ is separable.

Proof. See [Yosida, 1980, §V.4]. □

A strongly measurable $f : S \to X$ is called Bochner-integrable if there exists a sequence of simple functions f_n that converges a.e. to f, and such that the $\|f - f_n\|$ are integrable and:

$$\lim_{n\to\infty} \int_S \|f - f_n\| \, d\mu \to 0$$

A simple function is thus Bochner-integrable, and we can define its Bochner-integral in the obvious way, as the finite weighted sum of its values with the measure of each preimage as weight. For a Bochner-integrable function, we define its Bochner-integral as the limit of the integrals $\int_S f_n$, with f_n as above. One can show that this limit exists and does not depend on the choice of f_n. See [Yosida, 1980, §V.5]. From the definition, we see that the Bochner-integral (when it exists) is a Gelfand–Pettis integral.

Theorem B.15 (Bochner)**.** Let (S, μ) be a measure space, X a $\mathbb{K}$-Banach space and $f : S \to X$ a strongly measurable function.

1. f is Bochner-integrable iff $\|f\| : S \to \mathbb{R}$ is μ-integrable, in which case we have the triangle inequality:
$$\left\| \int_S f \right\| \leqslant \int_S \|f\|$$
2. If Y is a $\mathbb{K}$-Banach space, $f : S \to X$ is Bochner-integrable and $A \in L(X, Y)$, then $A \circ f$ is Bochner-integrable (in particular, strongly measurable) and
$$\int_S A \circ f = A \int_S f$$

Proof. See [Yosida, 1980, §V.5, Theorem 1, Corollary 1, Corollary 2]. Admitting all other statements, the last equality also follows because we know Bochner-integrals are Gelfand–Pettis integrals, which are unique in the case of Banach spaces. □

Because the Riemann and Bochner integrals are Gelfand–Pettis integrals, we conclude that when they both exist, they must be equal. (At least, when working with a locally convex space, so that Gelfand–Pettis integrals are unique when they exist.) We can thus simply speak of the 'integral' of a function, without ambiguity.

Remark B.16. We are not saying that when X is a Banach space, every Riemann-integrable function is Bochner-integrable. In fact, this is no longer true in the infinite-dimensional case.[13] The condition for Bochner-integrable functions to be separably-valued up to a null-set, turns out to be quite strong.

There is an ambiguity about integrability of functions that take value in a linear subspace: Let X be a topological vector space, Y a linear subspace and $f : S \to Y$ a function. If f is weakly integrable as a Y-valued function, then it is (by the general result about continuous linear maps remarked in the beginning) weakly integrable as an X-valued function, and the weak integrals coincide. What about the converse? We make the following elementary observations:

1. If Y is a closed linear subspace and f is Riemann-integrable as an X-valued function, it is also Riemann-integrable as a Y-valued function, simply by uniqueness of limits.

2. If X is a normed space, Y is any linear subspace and f is weakly integrable as an X-valued function, it is also -weakly integrable as a Y valued function: by Hahn–Banach, continuous linear functionals of Y extend to X.

B.4 Power series and meromorphy

Let X be a locally convex and quasi-complete topological $\mathbb{C}$-vector space. Then it has all Gelfand–Pettis contour integrals (B.12) and linear functionals separate points. We have seen (without proof) that weak and strong holomorphy are equivalent in this case. Much like the equivalence of weak and strong holomorphy reduces statements about holomorphic functions $U \to X$ to holomorphic functions $U \to \mathbb{C}$, the fact that the integral commutes with linear functionals allows to reduce many statements about vector-valued integrals to statements about integration of $\mathbb{C}$-valued functions:

Proposition B.17. 1. Change of variables in integration of differentiable functions $\mathbb{R} \to X$. In particular, contour integrals are well-defined.

2. Cauchy's integral theorem: if $f : U \to X$ is holomorphic, then its integral along closed contractible contours is 0.

3. Conversely, if f is continuous and the above holds, then f is holomorphic.

4. Homotopy invariance: if C_1, C_2 are homotopic contours in U and $f : U \to X$ holomorphic, then
$$\int_{C_1} f = \int_{C_2} f$$

5. Cauchy's integral formula:
$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(z_0,\delta)} \frac{f(z)}{z - z_0} dz$$

6. Cauchy's integral formula for derivatives:
$$f^{(n)}(z_0) = \frac{1}{2\pi i} \int_{B(z_0,\delta)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Here for Cauchy's integral representation for derivatives we also use that differentiation commutes with linear forms $X \to \mathbb{C}$.

Note also how the criterion for holomorphy by contour integration allows an alternative proof for the fact that the locally uniform limit of holomorphic functions is holomorphic. Also, Cauchy's integral

[13] Contrary to the case of real-valued functions, where every Riemann-integrable function is Lebesgue-integrable.

representation for the first derivative allows to generalize Hurwitz's theorem directly: by the triangle inequality for seminorms of Riemann-integrals in locally convex spaces, the locally uniform convergence $f_n \to f$ implies, together with Cauchy's integral formula, the uniform convergence $f_n' \to f'$. We also have:

Proposition B.18 (Laurent expansion)**.** Let $B'(z_0, R) \subseteq \mathbb{C}$ be a punctured disc, X as before and $f : B'(z_0, R) \to X$ a mapping. Then f is holomorphic iff there exist $(a_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}$ for which, pointwise,

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n \qquad (z \in B'(z_0, R))$$

in which case the convergence is uniform and absolute (for every continuous seminorm) in the annuli

$$\{r_1 < |z - z_0| < r_2\} \qquad (0 < r_1 < r_2 < R)$$

and the Laurent-coefficients a_n are then uniquely determined by

$$a_n = \frac{1}{2\pi i} \int_{\partial B(z_0,\delta)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Proof. The proof is the same as for complex-valued functions, with the absolute value replaced by a family of seminorms. Note that, while the uniqueness of the a_n and the formula for them can be deduced from the Hahn–Banach separation theorem and the complex-valued case, their existence cannot. □

We see that holomorphic functions can equivalently be defined as functions that are locally power series. We can define a meromorphic X-valued function as one with isolated singularities and whose Laurent-expansion has a finite singular part at every point. This can be checked on linear forms:

Proposition B.19. Let $f : U \to X$ be a function.

1. f is meromorphic iff $\lambda \circ f$ is meromorphic, for all $\lambda \in X^*$.

And if X is a topological algebra:

2. Meromorphic functions $U \to X$ form a ring.

If U is an open connected subset of $\mathbb{C}$, the reciprocal of a nonzero holomorphic function $U \to \mathbb{C}$ is meromorphic. Consider a Banach algebra X, and a holomorphic function $f : U \to X$ which is not identically zero. If $f(s)$ is invertible for s in a punctured neighborhood of (say) 0, then 0 is an isolated singularity of $1/f$. But $1/f$ need not be meromorphic at 0. Indeed: suppose $f(0)$ is not invertible, so that if $1/f$ is not holomorphic at s_0. If it is meromorphic, then $1/f(s) \sim (s - s_0)^{-N}A$ for some $N > 0$ and $A \in X - \{0\}$. In particular:

$$f(0)A = Af(0) = 0$$

But if in addition $f(0)$ is not a zero-divisor, this gives a contradiction. In (C.7), we give a sufficient condition for $1/f$ to be meromorphic.

B.5 Integration in function spaces

We come back to the question about holomorphy of functions $f : M \times U \to \mathbb{C}$, where M is a Riemannian manifold.

Proposition B.20 (Continuity of L^2-integrals)**.** Let M be an orientable Riemannian manifold. Let $I \subseteq \mathbb{R}$ be a compact interval. Let $f : M \times I \to \mathbb{C}$ be a function such that $f(\cdot, t) \in L^2(M)$ for all $t \in I$. Suppose that $t \mapsto f(\cdot, t)$ is continuous, so that it is integrable:

$$F(\cdot) = \int_I f(\cdot, t)dt \in L^2(M)$$

1. Let $N \subseteq M$ be a measurable subset. Then $t \mapsto f(\cdot, t)|_N \in L^2(N)$ is still integrable and

$$F(\cdot)|_N = \int_I f(\cdot, t)|_N dt \in L^2(N)$$

2. If f is (jointly) continuous, then $F(\cdot)$ is continuous (that is, it has a continuous representative) and

$$F(x) = \int_I f(x, t) dt \in \mathbb{C}$$

for all $x \in M$.

Proof. 1. Because integration commutes with continuous linear maps, in particular, the restriction to N.

2. While it is true that linear operators commute with integration, the evaluation map at x is ill-defined on L^2. We can extend it from $L^2(M) \cap C^0(M)$ using Hahn–Banach. But $C^0(M)$ cannot in any obvious way be mapped to $L^2(M)$, so we still need an argument why $F(\cdot)$ is continuous.

   Continuity is a local condition, so by the first statement we may assume M is compact. (Otherwise, replace M by a compact neighborhood of x.) In that case, we have a continuous inclusion map $i\colon C^0(M) \to L^2(M)$. Here, $C^0(M)$ is equipped with the L^∞ norm. Its image need not be closed. But $f : I \to C^0(M)$ is weakly integrable: it is continuous by uniform continuity of f. Hence when composed with i, we obtain the (weak) integral of f as a $L^2(M)$-valued function. We conclude that $F(\cdot)$ has a continuous representative. Finally, we can evaluate it in x by Hahn–Banach, as we remarked already. □

Proposition B.21 (Regularity of complex L^2-derivatives)**.** Let M be an orientable Riemannian manifold. Let $U \subseteq \mathbb{C}$ be open. Let $f : M \times U \to \mathbb{C}$ be a function such that $f(\cdot, s) \in L^2(M)$ for all s and $f : U \to L^2(M)$ is holomorphic.

1. If f is (jointly) continuous, then so is $f'(w, s)$.
2. If f is jointly (resp. separately) smooth, then so is $f'(w, s)$.

Proof. 1. Suppose f is (jointly) continuous. Fix $(w_0, s_0) \in M \times U$. We have by Cauchy's integral formula (B.17):

$$f'(\cdot, s_0) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\cdot, \zeta)}{(\zeta - s_0)^2} d\zeta \qquad \in L^2(M)$$

   where B is some small ball centered at s_0. By (B.20), we can evaluate this in w. For s in a small compact neighborhood V of s_0 and w in a compact neighborhood W of w_0, we have that $f(w, \zeta)/(\zeta - s)^2$ becomes uniformly continuous on $W \times V \times \partial B$, so the integral

$$f'(w, s) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(w, \zeta)}{(\zeta - s)^2} d\zeta$$

   defines a continuous function of (w, s).

2. Similarly, from Cauchy's integral formula. □

B.6 Holomorphy in function spaces

Continuing the remarks in the introduction to this section, we make the following elementary observations:

Proposition B.22 (Joint regularity implies C^k-smoothness)**.** Let U be an open set of $\mathbb{C} = \mathbb{R}^2$ and M a σ-compact Hausdorff topological space. We have the Banach space $C^0_b(M)$ of bounded continuous functions with the supremum norm, and the Fréchet space $C^0(M)$ of continuous functions with the topology of locally uniform convergence. Let $f : M \times U \to \mathbb{C}$ be jointly continuous. Denote the real coordinates on U by (t^1, t^2).

1. If the partial derivatives $\frac{\partial^{|\alpha|}}{\partial t^\alpha} f$ of all orders exist and are jointly continuous, then $f : U \to C^0(M)$ is smooth. If they are also bounded, then $f : U \to C^0_b(M)$ is smooth.

Now let M be a smooth manifold. We have the Fréchet space $C^\infty(M)$ with the topology of locally uniform convergence of all partial derivatives on compact subsets of coordinate neighborhoods.

2. Let f be jointly smooth. Then $f : U \to C^\infty(M)$ is smooth.

Proof. 1. Suppose those partial derivatives are jointly continuous, and let $K \subseteq M$ be compact. Let $s_0 \in U$ and write, for $(h,k) \in \mathbb{R}^2$ small and $w \in K$:

$$g(w, s_0, h, k) = f(w, s_0 + (h,k)) - f(w, s_0) - \frac{\partial f(w,\cdot)}{\partial t^1}(0) \cdot h - \frac{\partial f(w,\cdot)}{\partial t^2}(0) \cdot k$$

For f to be differentiable at s_0, we have to show that $g(w, s_0, h, k) = o(\|(h,k)\|)$ uniformly on K. By the mean value inequality:

$$\|g(w, s_0, h, k)\| \leqslant \|(h,k)\| \cdot \left\|\nabla g(w, s_0, \cdot)(\xi_{(h,k)})\right\|$$

for some $\xi_{(h,k)} \in \overline{B}(0, \|(h,k)\|)$. Because $\nabla g(w, s_0, 0) = 0$ and $\nabla g(w, s_0, \xi)$ is continuous in ξ, the RHS is $o_w(\|(h,k)\|)$. Moreover, it is jointly continuous in (w, ξ), and differentiable in ξ with continuous derivative. We use the mean value inequality once more and by compactness of K, we conclude that the RHS is $o_K(\|(h,k)\|)$, independently of $w \in K$. Thus $f : U \to C^0(M)$ is differentiable. By induction, it is smooth.

If all partial derivatives of $f : U \to C^0(M)$ are bounded, then we can take $K = U$ is the above proof, and the conclusion follows.

2. This follows from 1., because smoothness of $f : U \to C^\infty(M)$ is equivalent to smoothness of all partial derivatives $\frac{\partial}{\partial x^\alpha} f : U \to C^\infty(V) \subseteq C^0(V)$, for charts $(x^i) : V \to \mathbb{R}^{\dim M}$. ☐

Similarly one proves:

Proposition B.23 (Pointwise holomorphy iff C^k-holomorphy)**.** Let U be an open set of $\mathbb{C} = \mathbb{R}^2$ and M a smooth manifold.

1. If $f : M \times U \to \mathbb{C}$ is continuous, then it is pointwise holomorphic iff $f : U \to C^0(M)$ is holomorphic.

2. If for every chart (x^i) of M the partial derivatives $\partial^{|\alpha|} f(w,s)/\partial x^\alpha$ of any order exist and are continuous, then f is pointwise holomorphic iff $f : U \to C^\infty(M)$ is holomorphic.

Proof. That C^0-holomorphy implies pointwise holomorphy, is because evaluations are well-defined linear forms on C^0. The other direction is proved as for (B.22), by using the mean value inequality. That the regularity conditions on f can be weakened under the assumption of pointwise holomorphy, follows from (B.3). ☐

For convenience, we will say that f is C^0-smooth, C^0_b-smooth, C^0-holomorphic, C^∞-holomorphic, etc. We study the relation with L^2-holomorphy and pointwise holomorphy.[14] From now on, let M be a Riemannian manifold, $U \subseteq \mathbb{C}$ open and $f : M \times U \to \mathbb{C}$ a function.

Proposition B.24 (L^2-holomorphy implies pointwise holomorphy)**.** Let f be (jointly) continuous and such that $f(\cdot, s) \in L^2(M)$ for all s. Suppose $f : U \to L^2(M)$ is holomorphic. Then f is pointwise holomorphic with $f(w, \cdot)'(s) = f'(\cdot, s)(w)$ for all w, s.

Proof. Fix $w \in M$ and $s_0 \in U$. By assumption,

$$f(w,s) - f(w,s_0) = f'(w,s_0)(s - s_0) + (s - s_0)R(w,s) \qquad (s \to s_0)$$

[14]Most results will still hold with L^2 replaced by L^p, for $1 \leqslant p < \infty$.

where $R(w,s) = o(1)$ (in $L^2(M)$). By (B.21), $f'(w,s_0)$ is continuous in w. We want to look at this equation for fixed w and not just as an L^2 statement. We have $R(w,s_0) = 0$ almost everywhere, but that is not enough. W.l.o.g. suppose $s_0 = 0$. We may also assume $f(w,0) = f'(w,0) = 0$ for all w. For small s, by Cauchy's integral formula in L^2:

$$R(\cdot,s) = \frac{f(\cdot,s)}{s} = \frac{1}{2\pi i}\int_{\partial B}\frac{f(\cdot,\zeta)}{\zeta\cdot(\zeta-s)}d\zeta$$

for some ball B centered at $0 \in U$. By (B.20), this is continuous for fixed s, we can evaluate it in w and from the integral we then see that R is jointly continuous.
Now suppose $f(w,s)$ is not complex-differentiable at $s = 0$. Then there exists $\epsilon > 0$ and a sequence $s_n \to 0$ with $|R(w,s_n)| > \epsilon$. By joint continuity of $R(w,\cdot)$, we have $|R(w,0)| \geqslant \epsilon$ and by joint continuity there exists a neighborhood of $(w,0)$ on which $|R(z,s)| > \epsilon/2$. But then $\|R(\cdot,s)\|_{L^2}$ is bounded from below as $s \to 0$, contradiction. □

Proposition B.25 (L^2-holomorphy implies C^k-holomorphy)**.** Let f be jointly continuous and such that $f(\cdot,s) \in L^2(M)$ for all s. Suppose that f is L^2-holomorphic.

1. Then f is C^0-holomorphic and its C^0-derivatives coincide with the L^2-derivatives.
2. If for every chart (x^i) the partial derivatives $\partial^{|\alpha|}f(w,s)/\partial x^\alpha$ up to order n exist and are jointly continuous, then f is C^∞-holomorphic.

Proof. By (B.24), f is pointwise holomorphic. The two statements now follow from (B.23). □

Lemma B.26. Let $f : \overline{B}(0,R) \to \mathbb{C}$ be holomorphic and $0 < r < R$. Then

$$\sup_{s\in B(0,r)} |f(s)| \ll_{r,R} \int_{\partial B(0,R)} |f(z)||dz|$$

Proof. By Cauchy's integral formula, for $s \in B(0,r)$:

$$|f(s)| \ll \int_{\partial B(0,R)} \frac{|f(z)|}{|z-s|}|dz|$$

We conclude using $|z-s| \gg_{R,r} 1$ for $z \in \partial B(0,R)$. □

Proposition B.27 (L^2-holomorphy versus pointwise holomorphy)**.** Let $f : M \times U \to \mathbb{C}$ be continuous and such that $f(\cdot,s) \in L^2(M)$ for all s. Then the following are equivalent:

1. f is L^2-holomorphic.
2. f is pointwise holomorphic, and $s \mapsto \|f(\cdot,s)\|_2$ is locally bounded.
3. f is pointwise holomorphic, and $s \mapsto f(\cdot,s)$ is locally bounded by an L^2-function, independently of s.

Remark B.28. For the extra condition in the last statement, it suffices in particular that f is continuous with support contained in $T \times U$ for some compact $T \subseteq M$: it is then locally bounded by a function that is constant on T and 0 elsewhere.

Proof. $1 \implies 2$: Pointwise holomorphy is proven in (B.24). The local boundedness of the norms $\|f(\cdot,s)\|_2$ follows by continuity of $f : U \to L^2(M)$.

$2 \implies 1$: As a general fact, the local boundedness of $\|f(\cdot,s)\|_2$ implies that of $\|f'(\cdot,s)\|_2$: We have

$$f'(w,s) = \frac{1}{2\pi i}\int_{\partial B}\frac{f(w,z)}{(z-s)^2}dz$$

where B is a small ball centered at s, which we may assume of fixed radius $\delta > 0$ as long as s stays in a compact set. Then

$$\begin{aligned}\|f'(\cdot,s)\|_2^2 &= \int_M \left|\frac{1}{2\pi i}\int_{\partial B}\frac{f(w,z)}{(z-s)^2}dz\right|^2 dw\\ &\ll_\delta \int_M\int_{\partial B}|f(w,z)|^2|dz|dw\\ &= \int_{\partial B}\|f(\cdot,z)\|_2^2\,|dz|\\ &\ll_\delta 1\end{aligned}$$

by Cauchy-Schwarz and Fubini. In the last step we used that $z \mapsto \|f(\cdot,z)\|_2$ is locally bounded. Thus in particular, the higher derivatives $f^{(n)}(\cdot,s)$ are in L^2 and their norms are also locally bounded.

There is a converse. Suppose $f'(\cdot,s)$ has locally bounded norm, then

$$\begin{aligned}\|f(\cdot,s)-f(\cdot,s_0)\|_2^2 &= \int_M \left|\int_{[s_0,s]} f'(w,z)dz\right|^2 dw\\ &\leqslant \int_M |s-s_0|\int_{[s_0,s]}|f'(w,z)|^2|dz|dw\\ &= |s-s_0|\int_{[s_0,s]}\|f'(\cdot,z)\|_2^2\,|dz|\\ &\ll |s-s_0|^2 \sup_{z\in B(s_0,|s-s_0|)}\|f'(\cdot,z)\|_2^2\end{aligned}$$

where $[s_0,s]$ is a straight segment. Thus the local boundedness of $\|f'(\cdot,z)\|_2$ implies the continuity of $s\mapsto f(\cdot,s)$.

Now fix $s_0 \in U$. by subtracting from f the separately continuous L^2 function $(s-s_0)f'(w,s_0)$, we may assume that $f'(w,s_0)=0$.

From the computations above we successively have that $\|f''(\cdot,s)\|_2$ is locally bounded, that $\|f'(\cdot,s)\|_2$ is continuous at s_0 and that $f(\cdot,s)$ is differentiable at s_0.

$3 \implies 2$: Immediate.

$2 \implies 3$: Let $s_0 \in U$, and choose $0 < r < R$ such that $\overline{B}(s_0,R) \subset U$ and that $\|f(\cdot,s)\|_2$ is bounded on $\overline{B}(s_0,R)$. Let

$$g(w) = \sup_{s\in B(s_0,r)}|f(w,s)|$$

By (B.26) applied to all $f(w,\cdot)^2$:

$$\begin{aligned}\int_M g(w)^2dw &\ll_{r,R} \int_M\int_{\partial B(s_0,R)}|f(w,z)|^2|dz|dw\\ &= \int_{\partial B(s_0,R)}\|f(\cdot,z)\|_2^2\,|dz|\\ &\ll_R 1\end{aligned}$$

so that g is an L^2-function bounding all $f(\cdot,s)$ for $s \in B(s_0,r)$.

$3 \implies 1$: This follows of course from $3 \implies 2 \implies 1$, but we can give a direct argument.

So suppose $f(\cdot,s)$ is locally bounded by an L^2 function. Cauchy's integral formula shows that the same holds for $f'(\cdot,s)$, and the mean value theorem shows that it is also true for $(f(\cdot,s)-f(\cdot,s_0))/(s-s_0)$. By holomorphy at fixed w, we have:

$$f(w,s)-f(w,s_0) = f'(w,s_0)(s-s_0)+R(w,s) \qquad (s\to s_0)$$

where $R(w,s) = o_w(s - s_0)$. We can now apply dominated convergence to $R(w,s)/(s-s_0)$, and we obtain

$$\left\| \frac{R(w,s)}{s-s_0} \right\|_{L^2} \to 0 \qquad (s \to s_0)$$

which implies that $f : U \to L^2(M)$ is complex-differentiable at s_0. □

Remark B.29. We do not know[15] whether pointwise holomorphy, without any additional condition, implies L^2-holomorphy, for continuous $f : M \times U \to \mathbb{C}$ such that $f(\cdot, s) \in L^2(M)$ for all $s \in U$. We have a partial result: Because the L^2-norm is lower-semicontinuous, using the Baire category theorem one can show that $\|f(\cdot, s)\|$ is locally bounded in an open dense set, hence it is L^2-holomorphic in that open dense set.
We know that the implication does not hold with 'holomorphic' replaced by 'real analytic'. For example, the function

$$\begin{aligned} f : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x,s) &\mapsto \frac{xs}{1+(xs)^2} \end{aligned}$$

is jointly continuous, pointwise real analytic and in $L^2(\mathbb{R})$ for fixed s, but it is not real analytic as an $L^2(\mathbb{R})$-valued function: by a change of variables we have

$$\|f(\cdot, s)\|_{L_2} \asymp \frac{1}{|s|} \qquad (s \to 0)$$

So $f : \mathbb{R} \to L^2(\mathbb{R})$ is not even continuous at 0.

B.7 Meromorphy in function spaces

If X is (say) a Fréchet space, by definition a meromorphic function $f : U \to X$ is locally of the form $\frac{g(s)}{h(s)}$ with $g : U \to X$ and $h : U \to \mathbb{C}$ holomorphic, h not a zero divisor. We conclude that (with the same notations as before):

Proposition B.30. If $f : M \times U \to \mathbb{C}$ is C^0-meromorphic, then it is uniformly (pointwise) meromorphic.

A subtlety arises when we have no reason to assume that the numerator $g(w,s)$ is jointly continuous:

Proposition B.31 (L^2-meromorphy implies C^k-meromorphy)**.** Let $S \subseteq U$ be closed and discrete, $f : M \times (U - S) \to \mathbb{C}$ be continuous and L^2-meromorphic. Then it is C^0-meromorphic (and thus uniformly meromorphic). If in addition f is smooth in w on $M \times (U - S)$ with jointly continuous partial derivatives, then f is C^∞-meromorphic.

Proof 1. The question is local, so we may assume S is finite and the set T of L^2-poles of f is finite. Take a complex polynomial $h(s)$ such that $h(s)f(w,s)$ is L^2-meromorphic. Using what we know about L^2-holomorphic functions (B.25), it suffices to prove that $h(s)f(w,s)$ is continuous on $M \times U$ and not just on $M \times (U - (S \cup T))$, which is done in the lemma below. □

Proof 2. We have that f is C^0-holomorphic with isolated singularities. The Laurent-coefficients of those singularities can be expressed using integrals, and one sees that they are continuous in w. Hence when they are zero in L^2, they are zero a.e. in w, hence zero everywhere in w. □

Lemma B.32. Let $S \subseteq U$ be closed and discrete, $f : M \times U \to \mathbb{C}$ be L^2-holomorphic and $f : M \times (U - S) \to \mathbb{C}$ be continuous. Then f is continuous on $M \times U$. If in addition $f : M \times (U - S) \to \mathbb{C}$ is jointly smooth or smooth in w with jointly continuous partial derivatives, then f has that same property on $M \times U$.

[15] That is, we do not know whether it is known.

Proof. Using Cauchy's integral formula in L^2, we write $f(\cdot, s_0)$ in terms of the values $f(\cdot, s)$ for s on a small circle around s_0. By (B.20), we can evaluate the integral at points of M and the continuity at (w, s_0) follows from the continuity at (w, s) for s on that small circle. Any additional regularity properties follow from the same integral formula. ☐

From the second proof of (B.31) we also see that the order of the C^0-poles equals the order of the L^2-poles. Using the same technique, we prove a converse to (B.30):

Proposition B.33 (Uniform meromorphy implies C^k-meromorphy)**.** Let $S \subseteq U$ be closed and discrete, $f : M \times U \to \mathbb{C}$ uniformly meromorphic and $f : M \times (U - S) \to \mathbb{C}$ continuous.

1. Then f is C^0-meromorphic.
2. If in addition, $f : M \times (U - S) \to \mathbb{C}$ is smooth in w with continuous partial derivatives, then it is C^∞-meromorphic.

Proof. By (B.23), f is C^0- (resp. C^∞-) holomorphic with isolated singularities. We know that the Laurent-coefficients of isolated C^0-singularities are continuous in w. (This time, this is simply by definition of the space the holomorphic function takes values in.) If they are 0 at every point, then they are 0 in C^0 (resp. C^∞). We conclude that the isolated C^0-singularities are C^0-poles, and similarly for C^∞. ☐

Note that in (B.31), we haven't said that the L^2-poles or C^0-poles lie in S, and the two proofs we gave, do not clarify this. It is true: using the relations between all those notions of holomorphy we can reduce it to Riemann's theorem on removable singularities, for functions $U \to \mathbb{C}$:

Proposition B.34 (Removable singularities)**.** Let $f : M \times U \to \mathbb{C}$ be continuous. If f is L^2-meromorphic (resp. C^0-meromorphic) on $M \times U$, then it is L^2-holomorphic (resp. C^0-holomorphic).

Proof. Under the conditions from the statement, we know that L^2-meromorphy implies C^0-meromorphy, and that the L^2-orders of the poles are the same as their C^0-orders. Suppose s_0 is a pole. Because f is C^0-meromorphic, we can compose f with evaluation in each w and deduce that s_0 is a pole of some $f(w, \cdot)$. Fix such a w. Then $f(w, s)$ is meromorphic yet continuous in a neighborhood of s_0. Then s_0 is a removable singularity of $f(w, s)$, contradiction. ☐

C Fredholm integral equations

In this section, we let M be an oriented Riemannian manifold equipped with its canonical measure, and let $k : M \times M \to \mathbb{C}$ be a (measurable) *kernel*. We assume that k defines a bounded convolution operator on $L^2(M)$ by

$$K : g(x) \mapsto \int_M k(x,y)g(y)dy$$

This is the case if $k \in L^2(M \times M)$ is Hilbert–Schmidt (A.56) but it can also be bounded without being square-integrable (4.22). We denote by $\|K\|$ its operator norm, and by $\|k\|_2$ the L^2 norm when it is finite, which then equals the Hilbert Schmidt norm $\|K\|_2$. Recall that $\|K\| \leqslant \|K\|_2$. We study the *Fredholm equation* (of the second type)

$$(1 - \lambda K)g = f \tag{C.1}$$

where $\lambda \in \mathbb{C}$ and $f \in L^2(M)$. We seek to answer the questions:

1. When does (C.1) have a unique L^2 solution g?
2. Is g smooth when f is smooth?
3. Is the dependence of g on λ holomorphic? Meromorphic?

The operator $1 - \lambda K$ has a bounded inverse (by definition) when $\lambda = 0$ or λ^{-1} is not in the spectrum of K. This answers the first question. Observe that it suffices that $|\lambda| < \|K\|^{-1}$, in which case

$$(1 - \lambda K)^{-1} - 1 = \lambda K + \lambda^2 K^2 + \cdots \tag{C.2}$$

where the convergence is for the operator norm. This defines a holomorphic function of λ in the open disk $B(0, \|K\|)$. We cannot expect this identity to extend to larger values of λ, simply because $1 - \lambda K$ is usually not invertible for certain values of λ.
We will first discuss smoothness, and then present two approaches to study holomorphic and meromorphic dependence on λ, without restricting to the disk $B(0, \|K\|)$. The first, perhaps the most direct approach, is abstract in nature and relies on notions of differentiability in Banach spaces. It applies to all kernels that define a bounded operator. The second approach is due to Fredholm, who constructed an explicit meromorphic continuation of the inverse $(1 - \lambda K)^{-1}$ when K is a compact operator.

C.1 Regularity

If $\lambda \in \mathbb{C}$ such that $1 - \lambda K$ is invertible, the Fredholm equation (C.1) has a unique L^2 solution

$$g = (1 - \lambda K)^{-1} f$$

for all $f \in L^2$. The hope is that when f has nice properties, then so does g. In general, applying a (reasonable) smooth kernel to a smooth function yields a smooth function. What's special about $1 + \lambda K$, a 'perturbation' of the identity, is that the converse holds:

Theorem C.3 (Smoothness of the solution)**.** Let k and f be as above but λ arbitrary. Let g be any L^2 solution to $(1 - \lambda K)g = f$. Suppose that k is of class C^n ($n \geqslant 0$) and that it is compactly supported in the sense of (3.12)(4): for every compact $V \subseteq M$, the restriction $k : V \times M \to \mathbb{C}$ has compact support. Suppose also that $\mathrm{vol}(M) < \infty$. Then f is of class C^n iff g is of class C^n.

Proof. We have

$$f(x) = g(x) - \lambda \int_M k(x,y)g(y)dy$$

The key is that the second term is always of class C^n. Indeed, we fix $x_0 \in M$ and we restrict our attention to a compact neighborhood V of x_0 contained in a coordinate chart. Then the support of the integrand is contained in some compact subset of $M \times M$, independently of $x \in V$. Because M has

finite volume, $g \in L^2$ implies $g \in L^1$ by Cauchy-Schwarz. Because the integrand has compact support and k is of class C^n, its derivatives w.r.t. x up to order n are uniformly bounded by the integrable function $R \cdot |g(y)|$ for some $R > 0$ independent of $x \in V$. By the dominated convergence theorem, we conclude that the second term is of class C^n.
Thus f and g differ by a C^n function, and the conclusion follows. □

Note how the parameter λ has no relevance in the above result; we could absorb it in the kernel k. It is just there for the presentation.

C.2 The Fredholm equation for bounded operators

We now let λ and f depend holomorphically on some variable s, and prove that g depends also holomorphically on s. While it may seem more natural to write our family of operators as $\mu(s) - K$, the problem is that when $\mu(s_0) = 0$ this can not be seen as a 'perturbation of the identity', and (C.3) does no longer guarantee that the solutions $g(\cdot, s_0)$ are continuous when f and k are.

Lemma C.4 (Holomorphic operators and holomorphic functions)**.** Let X, Y, C be complex Banach spaces and $U \subseteq C$ open. Let $A : U \to L(X, Y)$ and $f : U \to X$ be holomorphic at $s_0 \in U$. Then $Af : U \to Y$ is holomorphic at s_0 with derivative

$$(Af)'(s_0) = A'(s_0)f + A(f'(s_0))$$

Proof. Analogous to the proof of the product rule. If we don't want to repeat the proof, we can in fact deduce it from the product rule by making $L(X, Y) \oplus X$ into a Banach algebra by defining

$$(A, x)(B, w) = (AB, Aw + Bx)$$

with submultiplicative norm $\|(A, x)\| := \|A\| + \|x\|$, so that $L(X, Y)$ and X embed isometrically into this space. □

Theorem C.5 (Holomorphy of the solution)**.** Let M and k be as in (C.3): M has finite volume and k has compact support in the sense that for V compact, $k : V \times M \to \mathbb{C}$ has compact support. Suppose in addition that k is continuous. Let $U \subseteq \mathbb{C}$ be open and $\lambda : U \to \mathbb{C}$ a holomorphic function such that all operators $1 - \lambda(s)K$ are invertible. Let $f : M \times U \to \mathbb{C}$ be continuous and supported in $T \times U$ for some compact $T \subseteq M$. So $f(\cdot, s) \in L^2(M)$ for all $s \in U$, and we can define

$$g(\cdot, s) = (1 - \lambda(s)K)^{-1} f(\cdot, s) \quad \in L^2(M)$$

By (C.3), the assumptions on k, $\mathrm{vol}(M)$ and the continuity of f imply that $g(\cdot, s)$ is continuous for all s, and in particular we can evaluate it in $w \in M$. We then have:

1. The following are equivalent:
   (a) f is pointwise holomorphic
   (b) f is L^2-holomorphic
   (c) g is L^2-holomorphic
   (d) g is pointwise holomorphic and jointly continuous
2. If the above equivalent statements hold and k is smooth, then f is jointly smooth iff g is jointly smooth.[16]

Proof. 1. We show that $(a) \implies (b) \implies (c) \implies (d) \implies (a)$:

$(a) \implies (b)$: Follows from the continuity of f and the condition on its support (B.27).

[16]Note that, under the assumption of pointwise holomorphy, joint smoothness can be formulated in terms of the partial derivatives w.r.t. w only (B.3)(3) and that joint smoothness implies C^∞-holomorphy (B.23).

(b) $\implies$ (c): This is elementary. Because $\lambda(s)$ is holomorphic, so is the the function $B(s) = 1 - \lambda(s)K$, which takes values in the Banach algebra of bounded operators on $L^2(M)$. By assumption, the operators $1 - \lambda(s)K$ are invertible, so $A(s) := B(s)^{-1}$ is holomorphic in s. By (C.4), the function

$$g = Af : U \to L^2(M)$$
$$s \mapsto A(s)f(\cdot, s)$$

is holomorphic.

(c) $\implies$ (d): For holomorphy for fixed w, we use (B.27): it suffices that $g(w, s)$ is jointly continuous. Fix (w_1, s_1) and let (w_2, s_2) vary in a compact neighborhood of (w_1, s_1). We write

$$\left|\int_M k(w_1, y)g(y, s_1) - \int_M k(w_2, y)g(y, s_2)\right|$$
$$\leqslant \int_M |k(w_1, y) - k(w_2, y)||g(y, s_1)|dy + \int_M |k(w_2, y)||g(y, s_1) - g(y, s_2)|dy$$

When $(w_2, s_2) \to (w_1, s_1)$, the first term approaches 0 by the dominated convergence theorem, where we use that k has compact support and that $g(\cdot, s_1) \in L^2 \subset L^1$. For the second term we have

$$\left|\int_M k(w, y)\left(g(y, s_1) - g(y, s_2)\right) dy\right| \leqslant \int_M |k(w, y)| \cdot |g(y, s_1) - g(y, s_2)|dy$$
$$\leqslant R\,\mathrm{vol}(M)\, \|g(\cdot, s_1) - g(\cdot, s_2)\|_{L^1}$$

where $R = \max_{y \in M} |k(w, y)|$ is finite because $k(w, \cdot)$ has compact support. By Cauchy-Schwarz and because $\mathrm{vol}(M) < \infty$,

$$\|g(\cdot, s_1) - g(\cdot, s_2)\|_{L^1} \ll \|g(\cdot, s_1) - g(\cdot, s_2)\|_{L^2}$$

Because $g : U \to L^2(M)$ is holomorphic, it is in particular continuous, and the we conclude that the second term goes to 0.

(d) $\implies$ (a): Fix w. It suffices to show that the integral

$$\int_M k(w, y)g(y, s)dy \tag{C.6}$$

defines a holomorphic function. Therefore it suffices to show that the integrand is bounded by an integrable function locally in s independently of s. Because k has compact support and g is jointly continuous, we can take a constant function.

2. Once again, it suffices to show that the integral (C.6) is jointly smooth. We want apply dominated convergence to switch the order of differentiation and integration. Because k is smooth, it remains to argue that $g^{(n)}(y, s)$ is jointly continuous for all n, the derivative being with respect to s. This follows from the joint continuity of g, and by repeatedly applying (B.3). □

Suppose that $(1 + \lambda(s)K)^{-1}$ has isolated singularities in U, which is the case when the spectrum of K is discrete, possibly with the exception of 0 as an accumulation point. Then the above result says that when f is (pointwise, say) holomorphic in U, then g is holomorphic with the exception of isolated singularities at those s for which $\lambda(s)^{-1}$ is in the spectrum of K. We are interested in meromorphy of g, and thus we need to know when the isolated singularities of $(1 + \lambda K)^{-1}$ are poles.

Theorem C.7. Let K be a compact operator on a complex Banach space. Then every nonzero point of its spectrum is a pole of its resolvent

$$R(s, K) = (K - s)^{-1}$$

Sketch of proof. The proof uses holomorphic functional calculus to show that the restriction of K to the range of the residue $E(\lambda)$ at an eigenvalue $\lambda \neq 0$ is compact, invertible and has spectrum $\{\lambda\}$. Thus this range is finite-dimensional, and by linear algebra in finite dimension over $\mathbb{C}$, there exists n with $(K-\lambda)^n E(\lambda) = 0$. By investigating the Laurent-coefficients, this says precisely that λ is a pole of order at most n. □

Corollary C.8 (Meromorphy of the solution)**.** Let all variables be as in (C.5), except for K, which we assume to be a compact operator and λ which we assume nowhere constant. Then:

1. If f is pointwise holomorphic, g is uniformly meromorphic, and even C^0-meromorphic and L^2-meromorphic.

2. If f is in addition (jointly) smooth, then g is C^∞-meromorphic.

Proof. 1. If f is pointwise holomorphic, then $g = (1+\lambda(s)K)^{-1} f$ is L^2-meromorphic because f is L^2-holomorphic and $(1+\lambda(s)K)^{-1}$ is meromorphic. Because g has isolated singularities and is continuous outside of them, we know from (B.31) that it is C^0-meromorphic. In particular, it is uniformly meromorphic.

2. Because g is C^0-meromorphic and its Laurent-coefficients are automatically smooth. □

C.3 Fredholm theory for compact operators

If K is Hilbert–Schmidt and $\lambda < \|K\|_2 < \infty$, then (C.2) is also convergent for the Hilbert–Schmidt norm. Recall that Hilbert–Schmidt operators form a Hilbert space (A.54) so that the limit will be Hilbert–Schmidt.

We thus make the following elementary observation:

Proposition C.9. If K is Hilbert–Schmidt and $\lambda < \|K\|_2$, the inverse does not only exist but is also given by a Hilbert–Schmidt integral operator:

$$(1-\lambda K)^{-1} f(x) = f(x) + \lambda \int_M \sum_{j=1}^{\infty} \lambda^{j-1} k_j(x,y) f(y) dy \tag{C.10}$$

where the iterated kernels k_j are given by $k_1 = k$ and

$$k_j(x,y) = \int_M k(x,z) k_{j-1}(z,y) dz \qquad (j \geqslant 2)$$

The hope is to meromorphically continue the kernel $R_\lambda = \sum_{j=1}^{\infty} \lambda^{j-1} k_j(x,y)$. Using the same notation for the associated integral operator, we have

$$(1-\lambda K) R_\lambda = K \tag{C.11}$$

When M is, say, the interval $[0,1]$, one can discretize the linear equation $(1+\lambda K)g = f$, by replacing K by the $(n+1)\times(n+1)$ matrix $(K_{ij}) = (K(\frac{i}{n}, \frac{j}{n}))_{0 \leqslant i,j \leqslant n}$, replacing f by the column vector $(f(\frac{i}{n}))_{0 \leqslant i \leqslant n}$ and similarly for g. By applying Cramer's rule to this linear system and taking the limit $n \to \infty$, sums turn into integrals, and Fredholm obtained, for a general finite-volume Riemannian manifold M:

Theorem C.12. Suppose $\mathrm{vol}(M) < \infty$ and that k is continuous and bounded. There exists an entire function, the Fredholm determinant $D(\lambda)$, and an entire function $D_\lambda(\cdot,\cdot)$ with values in the Banach space of continuous bounded kernels, such that

$$(1-\lambda K) \circ D_\lambda = D(\lambda) \cdot K \tag{C.13}$$

Moreover, when k is smooth, $D_{(\cdot)}(\cdot,\cdot)$ is jointly smooth.

Proof. See e.g. [Iwaniec, 2002, Appendix A.4]. The statement about smoothness is not mentioned there, but can be seen from the explicit formula given for the Taylor coefficients of $D_\lambda(x,y)$: applying a differential operator L to those coefficients can make them larger, but they still satisfy a bound of the form $\ll (\sqrt{m}C_L(x,y))^m/m!$, where m is the index, for some function C_L (depending on L) which can be taken constant locally in (x,y). □

Comparing this with (C.11) we obtain, still under the assumption that $\mathrm{vol}(M) < \infty$:

Corollary C.14. $D(\lambda)^{-1}D_\lambda$ defines a meromorphic continuation of R_λ, which is jointly smooth away from poles of D. In particular, $1 - \lambda K$ is invertible when λ is not a pole of D.

Corollary C.15. Let $\mathrm{vol}(M) < \infty$, $\lambda : \mathbb{C} \to \mathbb{C}$ be entire and $f : M \times \mathbb{C} \to \mathbb{C}$ continuous and pointwise holomorphic, with $f(\cdot, s) \in L^2$ for fixed s. Let k be a bounded kernel on M, compactly supported in the sense that for compact $V \subseteq M$, the restriction $k : V \times M \to \mathbb{C}$ has compact support. Then:

1. For $s \in \mathbb{C}$ not a pole of $D \circ \lambda$, there exists a unique solution $g(\cdot, s)$ to the Fredholm equation

$$(1 - \lambda(s)K)g(\cdot, s) = f(\cdot, s)$$

   (which is automatically continuous, by (C.3)).

2. We have
$$g(w, s) = f(w, s) + \frac{\lambda(s)}{D(\lambda(s))} \int_M D_{\lambda(s)}(w, y)f(y, s)dy$$

3. $g(w, s)$ is meromorphic for fixed w. More precisely, it is a holomorphic function divided by $D(\lambda(s))$.

4. If k is smooth and either:

   (a) f has support contained in $T \times \mathbb{C}$ for some compact $T \subseteq M$

   (b) M has a global chart (x^i) in which the derivatives of k are still bounded kernels, and $f(\cdot, s)$ is locally bounded by an L^2 function independent of s

   then f is jointly smooth iff g is.

Proof.
1. Because $1 - \lambda K$ is invertible when λ is not a pole of D.

2. We have $(1 - \lambda K)^{-1} = 1 + \lambda R_\lambda$ for small λ, so that indeed $1 + \lambda D_\lambda D(\lambda)^{-1}$ is the meromorphic continuation of $(1 - \lambda K)^{-1}$.

3. Because $s \mapsto D_{\lambda(s)}$ is holomorphic, it is in particular bounded by a constant function locally in s, so that the integral defines a holomorphic function for fixed w.

4. Once again, it suffices to show that the integral defines a jointly differentiable function. If the integrand has uniformly compact support, which is the case if (a) holds, then we can freely choose the order if integration and differentiation, and we are done. If (b) holds, then investigating the Taylor coefficients of D_λ shows that its derivatives are still bounded kernels. Together with the uniform integrability condition on f this allows us to differentiate inside the integral. □

Note how the meromorphic continuation of the kernel R_λ gives us meromorphy for free, while with the previous method we had to invoke holomorphic functional calculus to prove that the isolated singularities are indeed poles.

D Riemannian geometry

The below can be found in [Lee, 1991, Chapters 1–6].

Notation D.1. Let M be a smooth manifold. Its tangent bundle will be denoted TM, vector fields on M are global sections of the tangent bundle and form a module $\Gamma(TM)$ over $C^\infty(M)$. Smooth 1-forms are global sections of the dual TM^*, and form a $C^\infty(M)$-module $\Omega^1(M) = \Gamma(M, \Omega^1)$. It contains the differentials df of smooth real-valued functions $f \in C^\infty(M)$.

D.1 Riemannian manifolds

Definition D.2. A Riemannian metric on M is a smooth global section g of the vector bundle $(TM \otimes TM)^* = TM^* \otimes TM^*$ such that for all $p \in M$, $g(p)$ is a positive definite symmetric bilinear form on the tangent space T_pM. A *pseudo-Riemannian metric* is one which is not necessarily positive definite. An *isometry* between Riemannian manifolds (M, g), (N, h) is a diffeomorphism σ for which the pullback $\sigma^* h = g$. That is, such that if $p \in M$ with $\sigma(p) = q$ then $g_p(X, Y) = h_q(d\sigma|_p X, d\sigma_p Y)$ for tangent vectors $X, Y \in T_pM$. The isometry group is denoted $\mathrm{Isom}(M)$.

Example D.3 (Submanifolds)**.** A submanifold of a Riemannian manifold inherits a metric by restricting the metric to the tangent space of the submanifold. The restriction of a positive definite symmetric bilinear form to a subspace has indeed the same properties.

Example D.4 (Covering maps)**.** Let $\pi : E \to B$ be a covering map between manifolds, or more generally a local diffeomorphism. If B has a Riemannian metric h, we can locally pull it back by π and obtain a Riemannian metric on E. In particular, the universal cover of B has a canonical structure of a Riemannian manifold. Conversely, suppose E has a Riemannian metric g and π is a Galois cover, i.e. its automorphism group $\mathrm{Aut}(\pi)$ acts transitively on fibers. Suppose also that $\mathrm{Aut}(\pi) \subseteq \mathrm{Isom}(E)$. Then we can locally pushforward g to B in a well-defined way.

Example D.5 (Euclidean space)**.** The Euclidean space $\mathbb{R}^n$ is a Riemannian manifold with metric $\sum (dx^i)^2$ defined in the standard (global) chart.

Example D.6 (Spheres)**.** Let $R > 0$. Inside $\mathbb{R}^{n+1}$ the n-sphere S^n_R inherits a Riemannian metric from $\mathbb{R}^n$, by (D.3).

Example D.7 (Hyperbolic space)**.** Let $R > 0$. Hyperbolic n-space $\mathbf{H}^n_R$ can be defined in the following equivalent ways:

(a) (Hyperboloid model) As the upper sheet $\{\tau > 0\}$ of the hyperboloid $\tau^2 - |\xi|^2 = R^2$ in $\mathbb{R}^n \times \mathbb{R}$ which inherits the pseudo-Riemannian *Minkowski metric*

$$\sum_i (d\xi^i)^2 - (d\tau)^2$$

from $\mathbb{R}^{n+1}$, as in (D.3).

(b) (Poincaré ball model) As the open ball $B(R)$ in $\mathbb{R}^n$ with the metric in the standard chart given by

$$4R^4 \frac{\sum_i (du^i)^2}{(R^2 - |u|^2)^2}$$

(c) (Poincaré half-space model) As the upper half-space $\mathbf{H}^n_R = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ with coordinates $((x^i), y))$ and metric

$$R^2 \frac{\sum_i (dx^i)^2 + (dy)^2}{y^2}$$

Proof of equivalence. In [Lee, 1991, Proposition 3.5] it is shown by direct computation that the hyperbolic stereographic projection gives an isometry between the hyperboloid and the Poincaré ball: it sends a point P on the upper sheet to the unique intersection point of the segment $[P, Q]$ with $B(R) \subseteq \mathbb{R}^n = \mathbb{R}^n \times \{0\}$, where $S = (0, -R)$ is the hyperbolic south pole.
To compare the ball model and the half-space model, one shows that for $n = 2$ with the standard complex structure on these varieties, the map

$$w \mapsto -iR\frac{w + iR}{w - iR}$$

defines a biholomorphism that respects the metric. In real coordinates (u, v) on $B(R)$ it takes the form

$$(u, v) \mapsto \left(\frac{2R^2 u}{|u|^2 + (v - R)^2}, R\frac{R^2 - |u|^2 - v^2}{|u|^2 + (v - R)^2}\right)$$

with inverse

$$(x, y) \mapsto \left(\frac{2R^2 x}{|x|^2 + (y + R)^2}, R\frac{|x|^2 + y^2 - R^2}{|x|^2 + (y + R)^2}\right)$$

One then checks that these smooth maps are still inverses of each other when u and x take values in $\mathbb{R}^{n-1}$, and that they still preserve the metric. □

Remark D.8. With the Poincaré ball model, homotheties do induce diffeomorphisms between balls $B(R_1)$, $B(R_2)$ of different radius, but they are not isometries. Likewise for spheres.

D.2 Connections

Definition D.9. Let M be a smooth manifold and E be a vector bundle over M. A *connection* in E is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

denoted $\nabla(X, Y) = \nabla_X Y$, the *covariant derivative of* Y *in the direction of* X such that:

1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X.
2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y.
3. We have the product rule:
$$\nabla_X(fy) = f\nabla_X Y + (Xf) \cdot Y$$
for $f \in C^\infty(M)$.

A *linear connection* is a connection in TM.

Proposition D.10. For a linear connection ∇ in a vector bundle E at a point p:

1. $\nabla_X Y|_p$ depends only on the value of X at p and the values of Y in a neighborhood of p.
2. For a linear connection and a smooth curve with tangent vector X in a neighborhood of p, it depends only on the values of Y along the curve.

Proof. See [Lee, 1991, Lemma 4.1, Lemma 4.2, Exercise 4.7]. □

Definition D.11. Let γ be a smooth curve in M defined on some open interval I. A vector field along γ is a smooth map $I \to TM$ assigning to each $t \in I$ a tangent vector at $\gamma(t)$. Let $\Gamma(\gamma)$ be the set of vector fields along γ.

A vector field V along a curve can always be locally extended to a vector field on an open set of M whose restriction to γ is (locally) V.

Example D.12. The derivative $d\gamma/dt$ is a vector field along γ. More generally, $f(t)d\gamma/dt$ is one. If f is an immersion (i.e. df is injective at every point) then every vector field along γ is of this form.

Definition D.13 (Covariant derivative along a curve)**.** Let ∇ be a linear connection on M and γ be a smooth curve in M defined on some open interval, whose image . The *covariant derivative along* γ is the map

$$D_t : \Gamma(\gamma) \to \Gamma(\gamma)$$

defined by

$$(D_t V)(s) = \nabla_{\dot\gamma(s)} \widetilde{V}$$

where $\widetilde{V}$ is a smooth extension of V to a neighborhood of $\gamma(s)$.

Proposition D.14.

1. D_t is linear over $\mathbb{R}$.
2. It satisfies the product rule $D_t(fV) = \dot{f}V + fD_tV$.

Proof. See [Lee, 1991, Lemma 4.9]. □

Example D.15. In $\mathbb{R}^n$, the *Euclidean connection* $\overline{\nabla}_X Y$ is defined by letting X act on the components Y^i of Y in the standard basis. Then differentiation of V along a curve γ means differentiating the $V^i \circ \gamma$.

The linear combination of connections by a partition of unity is again a connection, so every manifold admits a linear connection. ([Lee, 1991, Proposition 4.5])

D.3 Geodesics and parallel transport

Definition D.16 (Acceleration along a curve)**.** Let M be a manifold with a linear connection ∇. Let γ be a curve in M. The *acceleration* of γ is the vector field $D_t\dot\gamma$ along γ.

Definition D.17 (Geodesic)**.** Let M be a manifold with a linear connection. A *geodesic* with respect to ∇ is a curve with acceleration 0.

By uniqueness, we can consider *maximal geodesics*: those who are defined on a largest possible interval.

Example D.18. In $\mathbb{R}^n$ with the Euclidean connection from (D.15), the geodesics are the straight lines: those whose second derivative vanishes.

Proposition D.19 (Existence and uniqueness of geodesics)**.** Let M be a manifold with a linear connection. For any tangent vector V at any point p, there exists a neighborhood in which there is a unique geodesic with derivative V at p.

Proof. Using the theory of differential equations. See [Lee, 1991, Theorem 4.19]. □

Definition D.20. A vector field V along a curve γ is *parallel along* γ iff $D_tV = 0$.

Proposition D.21 (Existence and uniqueness of parallel transport)**.** For any curve γ on a manifold M, point $p = \gamma(t_0)$ and tangent vector V_0 at p, there is a unique parallel vector field along the whole of γ extending V_0 at t_0, called the *parallel transport* of V_0, denoted $t \mapsto P^t_{t_0} V_0$.

Proof. See [Lee, 1991, Theorem 4.11]. □

Thus a curve is geodesic iff the parallel transport of the derivative at some point of the curve, equals the derivative at all other points.

D.4 Geodesics on Riemannian manifolds

From a linear connection on M, there is a natural way to obtain a connection on each $TM^{\otimes a}(TM^*)^{\otimes b}$, see [Lee, 1991, Lemma 4.6]. In particular we can consider the connection on $TM \otimes TM^*$ and apply the connection to a Riemannian metric considered as a section of that bundle.

Definition D.22. Let M be a Riemannian manifold with metric g. A linear connection ∇ on M is *compatible* with g if the following equivalent conditions hold:

1. $\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ for all vector fields X, Y, Z.
2. $\nabla g = 0$, with the action of ∇ defined as above.
3. For vector fields V, W along a curve:
$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle$$
4. For parallel vector fields along a curve, $\langle V, W\rangle$ is constant.
5. For any curve γ, parallel transport P_t^s is an isometry between the tangent spaces at $\gamma(t)$ and $\gamma(s)$.

Proof of equivalence. See [Lee, 1991, Lemma 5.2]. □

Definition D.23. Let M be a manifold. A linear connection ∇ is *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Proposition D.24. Let M be a Riemannian manifold. Then it has a unique compatible symmetric linear connection, the *Riemannian connection* or *Levi-Civita connection*.

Proof. See [Lee, 1991, Theorem 5.4]. □

On Riemannian manifolds we will always work with the Riemannian connection and the derived notions of (Riemannian) geodesics and parallel transport.

Example D.25. On $\mathbb{R}^n$ with the standard metric $\sum(dx^i)^2$, the Euclidean connection $\overline{\nabla}$ is compatible (this is Leibniz' rule) and symmetric (because the coordinate-wise definition of $\overline{\nabla}$ gives precisely the Lie bracket).

Proposition D.26. An isometry between two Riemannian manifolds takes geodesics to geodesics.

Proof. Because everything we defined so far is functorial. See also [Lee, 1991, Proposition 5.6]. □

D.4.1 The exponential map

Definition D.27. Let M be a Riemannian manifold and $p \in M$. The *exponential map* at p sends a tangent vector V to the point $\gamma_V(1)$ of the corresponding geodesic γ_V at time 1, if the geodesic is defined for that time.

Example D.28. For $M = \mathbb{R}$, the exponential at t sends s to $s + t$. The exponential on $T_0\mathbb{R}$ is thus very different from the Lie exponential.

Uniqueness implies $\gamma_{\lambda V}(1) = \gamma_V(\lambda)$ so that exp is defined on a star-shaped subset of T_pM centered around the origin.

Proposition D.29. The exponential map at $p \in M$ is smooth and restricts to a diffeomorphism between a neighborhood of $0 \in T_pM$ and one of $p \in M$, called a *normal neighborhood* of p.

Proof. See [Lee, 1991, Proposition 5.7, Lemma 5.10]. The first statement comes from the theory of differential equations, the second statement follows from the observation that, the differential of exp at 0 is the identity map of T_pM, as seen by letting $d\exp$ act on tangent vectors in terms of germs of curves, by composition with exp. □

Given an normal neighborhood U of a point, *normal coordinates* at the point are the components of $\exp^{-1} : U \to T_pM$ in an orthonormal basis. If we call them (x^i), the *radial distance function* is

$$r(x) = \sqrt{\sum_i (x^i)^2}$$

which is defined on U and the unit radial vector field

$$\frac{\partial}{\partial r} = \sum_i \frac{x^i}{r} \frac{\partial}{\partial x^i} = \frac{1}{r} \cdot \exp^{-1}$$

which is defined on $U - \{p\}$. They do not depend on the choice of normal coordinates (i.e. of an orthonormal basis of T_pM). The norm of $\partial/\partial r$ is 1, because by construction $(g_{ij}) = (\delta_{ij})$ at the point p, in normal coordinates.

Proposition D.30. With U as above, and $q \in U - \{p\}$, the vector $\partial/\partial r|_q$ is the velocity vector of the unit speed geodesic from p to q. That is, $t \mapsto \exp(t\partial/\partial r|_q)$

Proof. Let r be the radius of q. Then indeed, $\exp(r\partial/\partial r|_q) = q$, so that $\gamma_{\partial/\partial r|_q}(r) = q$. See also [Lee, 1991, Proposition 5.11]. □

D.4.2 Geodesics and distance

A curve (segment) defined on a closed bounded interval of $\mathbb{R}$ is one that extends smoothly to an open interval containing it. A curve is *regular* if its differential is injective, i.e. it is an immersion.

Definition D.31 (Length of a curve)**.** Let M be a Riemannian manifold and $\gamma : [a, b] \to M$ a smooth curve. Its *length* is the integral of the norm of its derivative from a to b. It is invariant under reparametrization.

Definition D.32. The Riemannian distance $d(p, q)$ between two points $p, q \in M$ is the infimum of lengths of piecewise regular (equivalently, regular) curve segments joining the two.

Proposition D.33. This defines a metric which induces the original topology of M.

Proof. That it defines a pseudometric follows from the triangle inequality. For the topology and positivity of the metric, see [Lee, 1991, Lemma 6.2]. □

This gives rise to the notions of *geodesic ball* and *geodesic sphere*.
One can consider smooth families of piecewise regular closed curves joining two points (with a common finite set of possibly non-smooth points) indexed by an open interval. We call a curve *critical* if the derivative of the length is zero at the curve, for every smooth family. A curve is *minimizing* if its length equals the geodesic distance between its endpoints. A curve is *locally minimizing* if every point of its interval of definition has a neighborhood to which the restriction is minimizing.
Using calculus of variations, one shows:

Proposition D.34. Every critical piecewise regular curve is in fact regular, and geodesic when we give it constant speed parametrization.

Proof. See [Lee, 1991, Theorem 6.6, Corollary 6.7]. Note that constant speed reparametrization is unique for regular curves. □

Note also that:

Proposition D.35. Geodesics have constant speed.

Proof. Follows by (D.22) of the compatibility of the Riemannian connection. See [Lee, 1991, Lemma 5.5]. □

Proposition D.36. Let $p \in M$ and q contained in a ball $\{r \leqslant \epsilon\}$ of a normal neighborhood of p, where r is the radial distance to p. Then the radial geodesic from (D.30) is the unique minimizing piecewise regular curve from p to q. Consequently, inside such a geodesic ball the radius r equals the geodesic distance to p, and every geodesic is locally minimizing.

Proof. See [Lee, 1991, Proposition 6.10, Corollary 6.11, Theorem 6.12]. □

Corollary D.37. Let $p \in M$ and U a normal neighborhood of p. Let r denote the geodesic distance to p on U. Then r^2 is smooth on U and r is smooth on $U - \{p\}$.

Proof. If $q \in U$ has normal coordinates (x^i) then

$$r^2 = \sum (x^i)^2$$

which is smooth in q. Next, at $q \neq p$ we have $r \neq 0$ so r too is smooth there. □

D.4.3 Completeness

Because geodesics have constant speed (D.35), they can only be reparametrized by homotheties and translations of the interval of definition, and the property of being defined on $\mathbb{R}$ is an intrinsic notion, which only depends on the image of the geodesic.

Definition D.38. A Riemannian manifold is *geodesically complete* if all geodesics can be defined on the whole of $\mathbb{R}$.

Theorem D.39 (Hopf–Rinow)**.** For a connected Riemannian manifold M, TFAE:

1. M is geodesically complete.
2. M has the Heine-Borel property: every closed bounded subset is compact.
3. M is complete as a metric space for the Riemannian distance.
4. There exists a point p for which the exponential is defined on the whole of T_pM.
5. Every two points can be joined by a (not necessarily unique) minimizing geodesic segment.
6. Every two points can be joined by a geodesic segment.

Proof. See [Lee, 1991, Theorem 6.13, Corollary 6.14, Corollary 6.15] and [Petersen, 2016, Theorem 5.7.1] for the second statement. □

Proposition D.40. An isometry ϕ between complete Riemannian manifolds X, Y is determined by the image of one point and the differential at that point.

Proof. Let $p, q \in X$ with p fixed. Let ϕ, ψ be isometries with equal differential at p. Let γ be a geodesic from p to q with initial velocity V. By (D.26) both $\phi \circ \gamma$ and $\psi \circ \gamma$ are geodesics with initial vector $d\phi|_p V$ joining $\phi(p)$ with $\phi(q)$ resp. $\psi(q)$. By uniqueness, they are equal. In particular their endpoints are equal, and $\phi(q) = \psi(q)$. □

Proposition D.41. Let M be a complete Riemannian manifold. TFAE:

1. All geodesics are minimizing.
2. All points are joined by a unique geodesic.
3. The exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for all $p \in M$.

Proof. See the Math Stackexchange post [*Manifolds with geodesics which minimize length globally* 2018]. □

D.5 Integration

Definition D.42. Let (M, g) be a Riemannian manifold with an orientation. Its associated *volume form* is the unique volume form that equals 1 on positive orthonormal bases. That is, $\omega = \sqrt{\det g} dx^1 \wedge \cdots dx^n$ in any chart $(x^1, \ldots, x^n)$, or more generally $\omega = \sqrt{|\det g|}\phi^1 \wedge \cdots \phi^n$ for any local frame $(E_1, \ldots, E_n)$ with dual $(\phi^1, \ldots, \phi^n)$. Integrating functions against this volume forms gives rise to the *Riemannian measure*.

This makes M into a measure space, which gives us a notion of integration of measurable functions, and not just of differential forms.

Example D.43.

1. For $\mathbb{R}^n$, we have $\omega = dx^1 \wedge \cdots \wedge dx^n$, and we get the Lebesgue measure.
2. For $\mathbf{S}^n_R$ we have $\omega = i_N(dx^1 \wedge \cdots \wedge dx^n)$ where N is the outward unit normal.
3. For $M = \mathbf{H}^{n+1}_R$ with the Poincaré half-space model, we have $\omega = (R/y)^{n+1} dx^1 \wedge \cdots \wedge dx^n \wedge dy$, with associated *hyperbolic measure* for $R = 1$.

D.6 The Laplace–Beltrami operator

The choice of a nondegenerate bilinear form g on a finite-dimensional real vector space V determines a isomorphism with its dual, which sends $v \mapsto g(v, \cdot)$. When the bilinear form g is clear from the context, we will denote the isomorphism by $\flat : v \mapsto v^\flat$, called *flat*. If (e_i) is a basis of V with dual basis (e_i^*), and g has matrix $A = (a_{ij})$ in this basis, then $\flat$ sends $\sum_i \lambda_i e_i$ to $\sum_{i,j} \lambda_i a_{ij} e_j^*$. The inverse map is denoted $\sharp$.
If g is a Riemannian metric on a manifold M, then to each vector field V on X we can associate its *flat* $V^\flat$ which is a 1-form. Indeed, the explicit formula above ensures that it is smooth. Likewise, a 1-form gives rise to a vector field by applying $\sharp$.

Definition D.44. Let M be a Riemannian manifold with volume form ω, let $f : M \to \mathbb{R}$ smooth and $X \in \Gamma(TM)$ a vector field.

- The *gradient* of f is the vector field $\operatorname{grad} f = (df)^\sharp$.
- The *divergence* of X is the unique smooth function $\operatorname{div} X$ for which the interior derivative $d(i_X \omega) = \operatorname{div} X \cdot \omega$.
- The *Laplace-Beltrami operator* (or simply *Laplacian*) of f is $-\Delta$ defined as:[17]

$$-\Delta f = -\operatorname{div}(\operatorname{grad} f)$$

Proposition D.45. In a chart $(x^1, \ldots, x^n)$ such that g has matrix (g_{ij}) in the basis $\left(\frac{\partial}{\partial x^i}\right)$ with inverse (g^{ij}), we have

$$-\Delta f = -\sum_{i,j} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right)$$

where $|g|$ is the determinant of the matrix (g_{ij}).

Proof. We have $df = \sum_j \frac{\partial f}{\partial x^j} dx^j$, hence $\operatorname{grad} f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$. We have

$$i_Y \left(\bigwedge_i dx^i \right) = \sum_j (-1)^{j+1} Y_j \bigwedge_{k \neq j} dx^k$$

and thus:

$$d(i_{\operatorname{grad} f}(\omega)) = d\left(\sqrt{|g|} \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-1}} \wedge \left(\sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \right) \right)$$

[17] Also defined without the minus sign. This way we obtain a positive operator; see (G.17).

$$= d\left(\sqrt{|g|}\sum_{i,j}(-1)^{j+1}g^{ij}\frac{\partial f}{\partial x^i}\bigwedge_{k\neq j}dx^k\right)$$

$$= \sum_{i,j}\frac{\partial}{\partial x^j}\left(\sqrt{|g|}\sum_{i,j}g^{ij}\frac{\partial f}{\partial x^i}\right)\bigwedge_k dx^k \qquad \square$$

Example D.46 (The Laplacian on hyperbolic spaces)**.** Let $n \in \mathbb{N}_{>0}$. Consider the hyperbolic space $\mathbf{H}^{n+1}$ as in (D.7)(c). We have from (D.45):

$$-\Delta = \begin{cases} -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) & : n = 1 \\ -y^2\left(\sum_i \frac{\partial^2}{\partial (x^i)^2}+\frac{\partial^2}{\partial y^2}\right) + (1-n)y\frac{\partial}{\partial y} & : n \geqslant 1 \end{cases}$$

D.7 Isometry groups

Theorem D.47 (Myers, Steenrod)**.** The isometry group G of a Riemannian manifold M is a (real) Lie group for the compact-open topology, and its action on M is smooth. If M is compact, then so is G.

Proof. See their original paper [Myers and Steenrod, 1939, Theorem 10]. $\square$

Note that a continuous group homomorphism between Lie groups is smooth, so that a Lie group structure, if it exists, is unique.
One can show that the connected component of G still acts transitively. This is a result from topology, which relies on the Baire category theorem:, if a group acts continuously and transitively on a locally compact Hausdorff space X that is σ-compact (countable union of compact sets) then the action of G on a fixed $x \in X$ is open. So if X is connected, open subgroups act transitively.

D.8 Stabilizers

Proposition D.48. Let M be a Riemannian manifold with isometry group G, $p \in M$ and K the stabilizer of p.

1. The group homomorphism $G \to O(T_pM) : \sigma \mapsto (d\sigma)_p$ is continuous.
2. K is closed in G.
3. If M is connected and complete, the homomorphism $G \to O(T_pM)$ is injective.

Proof. 1. Let U be an open geodesic ball with center p which is a normal neighborhood. Then the action of G on M restricts to U, and by functoriality of the compact-open topology, the restriction map $G \to \mathrm{Isom}(U)$ is continuous. For $\sigma \in K$ we have

$$\exp_p \circ d\sigma = \sigma \circ \exp_p$$

and again by functoriality of the compact-open topology, $\mathrm{Isom}(U) \to \mathrm{Diff}(\exp^{-1}(U)) : \sigma \mapsto d\sigma$ is a homeomorphism on its image. Such a $d\sigma$ extends uniquely to a linear map in $O(T_pM)$. It remains to show that taking the unique extension defines a continuous map. Note first that the norm topology on $\mathsf{GL}(T_pM)$ is the same as the compact-open topology: both are the topology of compact convergence. The restriction map $O(T_pM) \to \mathrm{Diff}(\exp^{-1}(U))$ is continuous by functoriality of the compact-open topology, and it is a homeomorphism on its image, because we can recover the norm of a linear operator by its action on a neighborhood of 0. That is, the image is a metric space and the restriction map is an isometry on its image. We conclude that the composition

$$\mathrm{Isom}(M) \to \mathrm{Isom}(U) \to \mathrm{Diff}(\exp^{-1}(U)) \to O(T_pM)$$

is continuous.

2. Because G has the compact-open topology.

3. From (D.40) an isometry is determined by its differential at p. □

Theorem D.49 (van Dantzig, van der Waerden)**.** Let M be a locally compact connected metric space with isometry group G, given the compact-open topology. Then G acts properly on M.

Proof. See [Dantzig and Waerden, 1928]. □

Proposition D.50 (Compactness of isotropy groups)**.** Let M be a Riemannian manifold and $p \in M$.

1. The isotropy subgroup K of p is compact.

2. Suppose M is complete. Then the identification from (D.48) has closed image and gives an isomorphism of Lie groups between K and a closed subgroup of $O(T_pM)$.

Proof.

1. Because the action $G \curvearrowright M$ is proper by (D.49).

2. An bijective continuous group homomorphisms between Lie group is automatically an isomorphism. It suffices to show that the identification is continuous. Its image will be closed (hence a Lie group) because K is compact. The continuity is precisely (D.48)(1). □

E Symmetries of manifolds

E.1 Isotropic manifolds

Definition E.1 (Isotropic manifold)**.** A Riemannian manifold is *isotropic* (at a point p) if the stabilizer (isotropy subgroup) of every point (resp. the point p) acts transitively on unit tangent vectors, with the action from (D.48).

Proposition E.2. Let M be a Riemannian manifold and $p \in M$ with stabilizer K. TFAE:

1. M is isotropic at p.
2. There exist arbitrarily small $\delta > 0$ such that K acts transitively on the geodesic sphere $S(p, \delta)$.
3. For all $\delta > 0$ for which the geodesic ball $B(p, \delta)$ is a normal neighborhood, K acts transitively on $S(p, \delta)$.

Proof. The exponential map at p commutes with the action of K: for $g \in K$:

$$\exp(dgV) = g \exp(V)$$

and if one side is defined, so is the other. □

E.2 Homogeneous spaces

Definition E.3 (Homogeneous Riemannian manifold)**.** A Riemannian manifold is *homogeneous* if its isometry group acts transitively on points.

Proposition E.4. A homogeneous Riemannian manifold is complete.

Proof. By (D.29), for a point p there exists $\delta > 0$ such that all geodesics through p are defined at time $[-\delta, \delta]$. By homogeneity and (D.26), we can take the same δ for all points. □

More generally we define:

Definition E.5. Let G be a Lie group. A manifold M together with a smooth and transitive action of G is a *homogeneous (G-)space*. A morphism of homogeneous G-spaces is a smooth map that respects the G-action.

A morphism of G-spaces is necessarily surjective.

Theorem E.6 (Construction of homogeneous spaces)**.** Let G be a Lie group and H a closed subgroup. The left coset space G/H has a unique differentiable structure for which the projection is a smooth submersion. Its dimension is $\dim G - \dim H$ and it is a homogeneous space for the action $g_1 \cdot (g_2 H) = (g_1 g_2)H$.

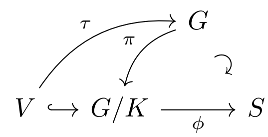
Proof. See [Lee, 2012, Theorem 21.17]. □

Theorem E.7. Every homogeneous G-space is of the above form, up to isomorphism. More precisly, the stabilizer K of a point p is a closed subgroup of G and the bijection from the orbit-stabilizer theorem provides the isomorphism.

Proof. See [Lee, 2012, Theorem 21.18]. □

Proposition E.8 (Local parametrization by a Lie group)**.** Let M be a homogeneous G-space and $x_0, y_0 \in M$. Then there exists an open neighborhood U of y_0 and a smooth embedding $\phi : U \to G$ such that $\phi(y)x_0 = y$ for all $y \in U$.

Proof. We may suppose $x_0 = y_0$: for arbitrary x_0 it suffices to replace $\phi(y)$ by $\phi(y) \cdot \sigma$ where $\sigma \in G$ is such that $\sigma x_0 = y_0$. Let K be the stabilizer of y_0. By (E.7), we have an isomorphism $\psi : G/K \xrightarrow{\sim} S$ as G-spaces and the projection $\pi : G \twoheadrightarrow G/K$ is a submersion. By the local normal form for submersions, there exists an open neighborhood V of $\pi(e) = [e] \in G/K$ and a smooth local section $\tau : V \to G$ of π which is an embedding.



Let $U = \phi(V)$ and $\psi = \tau \circ \phi^{-1}$. This does what we want: for $y \in U$ we have by G-equivariance of ϕ and $\phi([e]) = y_0$ that:

$$\begin{aligned}\tau(\phi^{-1}(y)) \cdot y_0 &= \phi(\tau(\phi^{-1}(y)) \cdot [e]) \\ &= \phi(\pi(\tau(\phi^{-1}(y)))) \\ &= \phi(\phi^{-1}(y)) \\ &= y\end{aligned}$$

□

E.3 Symmetric spaces

For a point p on a Riemannian manifold, we can take a geodesic ball B inside a normal neighborhood U (§D.4.1) which is stable by the *geodesic inversion* which sends $\exp(V)$ to $\exp(-V)$, equivalently, $\gamma_V(1)$ to $\gamma_V(-1)$ for $V \in \exp(B)$, equivalently, it sends small geodesics through p to the geodesics at the same speed in the other direction. It is smooth with differential $-\mathrm{id}$ at p. Nothing guarantees that it is an isometry: normal coordinates tell little about the metric at points other than the center.

Definition E.9. A *(Riemannian) locally symmetric space* is a connected Riemannian manifold for which the following equivalent conditions hold:

1. For every point, the geodesic inversion of any geodesic ball contained in a normal neighborhood is an isometry.

2. For every point, the geodesic inversion of some small geodesic ball contained in a normal neighborhood is an isometry.

3. For every point, there exists a local isometry defined on an open neighborhood that fixes the point and has differential $-\mathrm{id}$.

A *symmetric space* is one for which it extends to a global isometry.

Note that by (D.40), an isometry with differential $-\mathrm{id}$ at a fixpoint is uniquely determined by that property, on a complete manifold. By (D.29) and Hopf–Rinow (D.39), every Riemannian manifold is locally complete.

Proposition E.10. 1. A symmetric space is complete.

2. A symmetric or complete isotropic manifold is homogeneous.

Proof. 1. Take a point p and a geodesic γ defined on an interval $[-s, s]$. The geodesic reflection around $\gamma(s/2)$ defines, by uniqueness and by (D.26), an extension to the interval $[-s, 2s]$.

2. Take two points p, q and a geodesic γ joining the two. It has finite length and we may consider the geodesic inversion σ around its midpoint $\gamma(t_0)$, or in the isotropic case, any isometry fixing $\gamma(t_0)$ whose differential sends $\dot\gamma(t_0)$ to $-\dot\gamma(t_0)$. Applying this to γ and following the curve in the opposite direction, we obtain a geodesic with the same initial velocity as γ. Thus it coincides with γ on the intersection of their domains. Comparing lengths (which are preserved by isometries) we see that p and q are interchanged by σ. □

Every symmetric space is homogeneous and thus of the form G/K where K is the isotropy subgroup of a point, by (E.7).

F Differential operators

Notation F.1. Let M be a manifold. Let (x^i) be a chart defined on an open set U. For a multi-index $a \in \mathbb{N}^n$ we have the operator $D^a = \prod(\partial/\partial x^i)^{a_i}$ which acts on $C^\infty(U)$.

Definition F.2 (Differential operator)**.** A (smooth) differential operator D on a smooth manifold M is a linear operator on $C^\infty(M)$ for which the following equivalent conditions hold:

1. If (x^i) is a local chart on an open set U:
   - D followed by the restriction to U is a formal power series in the $\partial/\partial x^i$ with coefficients in $C^\infty(U)$ (applied to the restriction to U).
   - Locally around every point, only finitely many of the coefficients are nonzero functions (i.e. attain nonzero values). Equivalently, on every relatively compact open subset W of U with $\overline{W} \subset U$, only finitely many coefficients are nonzero. That is, it is a finite linear combination of the D^a with coefficients in C^∞.
2. D is a local operator: If ϕ and ψ coincide on an open set V, then so do $D\phi$ and $D\psi$.
3. For all smooth ϕ we have $\operatorname{supp}(D\phi) \subseteq \operatorname{supp}(\phi)$.
4. If ϕ has compact support, then so does $D\phi$ and $\operatorname{supp}(D\phi) \subseteq \operatorname{supp}(\phi)$.

Proof of equivalence. We clearly have $1 \implies 2 \implies 3 \implies 4$. For $4 \implies 1$, see [Helgason, 1984, Theorem II.1.4]. □

Way may state some results only for differential operators on real-smooth functions, but it should be noted that the analogous statements for complex differential operators hold as well.
They form an associative algebra under composition and pointwise addition with center $\mathbb{R}$, denoted $\mathrm{E}(M)$.

Definition F.3. Let $p \in M$. Denote $D|_p : C^\infty(M) \to \mathbb{R} : f \mapsto D(f)(p)$ for D composed with evaluation at p. The vector space of all these $\mathbb{R}$-linear forms on germs of functions at p is denoted $\mathrm{E}(M)_p$, the *differential operators at p*. See (F.12) for a motivation of the notation.

Proposition F.4 (Unique representation of differential operators)**.** Let M be a smooth n-dimensional manifold and U a coordinate neighborhood for a chart (x^i). let $D \in \mathrm{E}(M)$ be a differential operator.

1. There exists a unique formal power series in n variables with coefficients in $C^\infty(U)$ with locally only finitely many nonzero coefficients, that equals D when evaluated in the $\partial/\partial x^i$.
2. Let $p \in U$. Then there is a unique real polynomial $P(t_i)$ in n variables such that $D|_p = P(\partial/\partial x^i)|_p$. In particular, by sending $t_i \mapsto \partial/\partial x^i|_p$ we get a linear isomorphism

$$\operatorname{Sym}(T_pM) \xrightarrow{\sim} \mathrm{E}(M)_p \tag{F.5}$$

Proof. 1. The existence is by definition of a differential operator. For uniqueness, we obtain the coefficients of the power series by letting D act on monomials in the x^i. 2. For the exact same reason. □

Remark F.6.

1. The $\mathbb{R}$-linear isomorphism $\operatorname{Sym}(T_pM) \xrightarrow{\sim} \mathrm{E}(M)_p$ depends on the chart.
2. $\mathrm{E}(M)$ and $\operatorname{Sym}(T_pM)$ are algebras, but the composition $\mathrm{E}(M) \twoheadrightarrow \mathrm{E}(M)_p \xrightarrow{\sim} \operatorname{Sym}(T_pM)$ (defined by a chart) is only a morphism of vector spaces. Had it been a morphism of algebras, then $\mathrm{E}(M)$ would be commutative, which it clearly need not be.

Definition F.7 (Pushforward of a differential operator)**.** Let $\sigma : M \to N$ be a diffeomorphism. For $D \in \mathrm{E}(M)$ define the *pushforward* by the following two equivalent definitions:

1. For $f \in C^\infty(N)$, define $\sigma_* D \in \mathrm{E}(M)$ by $\sigma_* D(f) = D(f \circ \sigma) \circ \sigma^{-1}$.

2. For $f \in C^\infty(N)$ and $p \in M$, define $(\sigma_* D)|_{\sigma(p)}(f) = D|_p(f \circ \sigma)$.

Proof of equivalence. The two are equal because the second definition is obtained by evaluating the first at $\sigma(p)$. □

Note that $\sigma_* D$ is smooth from the first definition, and it decreases supports, so it is indeed a differential operator.

Proposition F.8 (Properties of the pushforward)**.** Let $\sigma : M \to N$ be a diffeomorphism.

1. σ_* sends C^∞-functions to C^∞ functions and vector fields to vector fields: it is the usual push-forward.

2. $(\sigma \circ \tau)_* = \sigma_* \circ \tau_*$ when $\tau : P \to M$ is another diffeomorphism.

3. σ_* is an isomorphism of algebras with inverse $(\sigma^{-1})_*$.

Thus the pushforward gives a (left) action of the diffeomorphism group $\mathrm{Diff}(M) \curvearrowright \mathrm{E}(M)$. One also denotes the *pullback* $\sigma^* D := (\sigma^{-1})_* D$ by D^σ; one then has the exponentiation rule $(D^\sigma)^\tau = D^{\sigma\tau}$. If $\phi = (\phi^i)$ is a chart, then $\phi_*(\partial/\partial\phi^i) = \partial/\partial x^i$, where (x^i) is the standard chart on open sets of $\mathbb{R}^n$.

F.1 Grading

Lemma F.9. Let $U \subseteq \mathbb{R}^n$ open, $f \in C^\infty(U)$ smooth and D^α, D^β monomials in the $\partial/\partial x^i$ of degrees a and b.

1. $D^\alpha f D^\beta$ is a polynomial in the $\partial/\partial x^i$ of degree $\leqslant a + b$ with coefficients in $C^\infty(U)$

2. $[D^\alpha, f]$ is a polynomial in the $\partial/\partial x^i$ of degree $< a$ with coefficients in $C^\infty(U)$.

3. More precisely, if $D^\alpha = \prod_j \partial/\partial x^{i_j}$ then

$$[D^\alpha, f] = \sum_j \partial f/\partial x^{i_j} \prod_{k \neq j} \partial/\partial x^{i_k} + P(\partial/\partial x^j)$$

   where P has degree $\leqslant n - 2$.

Proof. The statements are trivial for $a = 0$. Let $a > 0$ and write $D^\alpha = \partial/\partial x^i D^\gamma$ for some i.

1. By induction on a. For $a = 1$ this is true by the product rule. In the induction step, we apply the $a = 1$ case to all monomials appearing in $D^\gamma f D^\beta$.

2, 3 This is the product rule applied to $D^\alpha(fg)$. □

F.1.1 Global grading

Definition F.10 (Grading of differential operators)**.** A differential operator $D \in \mathrm{E}(M)$ is of degree at most $n \in \mathbb{N}$ if the following equivalent conditions hold:

1. In every chart $(U, (x^i))$, D is a polynomial of degree $\leqslant n$ in the $\partial/\partial x^i$ with coefficients in $C^\infty(U)$.

2. There exists a cover by charts $(U, (x^i))$ in which D is a polynomial of degree $\leqslant n$ in the $\partial/\partial x^i$ with coefficients in $C^\infty(U)$.

3. Define degree 0 operators to be $C^\infty(M)$ functions and define inductively D to have degree $\leqslant n$ if $[D, f]$ has degree $< n$ for all $f \in C^\infty(M)$.

Note that degree $\leqslant n - 1$ according to the third definition implies degree $\leqslant n$ by (F.9)(1).

Proof of equivalence. 1 $\implies$ 2: Ok. 2 $\implies$ 3: Degree 0 elements according to definition 2 are indeed C^∞-functions. The condition in 3. can be verified inductively and locally, and we conclude using (F.9)(2). 3 $\implies$ 1: A C^∞ function indeed has degree 0 according to definition 1. Suppose definition 3 implies definition 1 up to degree $n-1$. If $[D, f]$ has degree $< n$ for all $f \in C^\infty(M)$, it is a polynomial of degree $< n$ in any chart $(U, (x^i))$. Let $D = \sum_\alpha g_\alpha D^\alpha$ in this chart. Let $m = \dim M$ and denote $D^\alpha = \prod_{i=1}^m (\partial/\partial x^i)^{\alpha_i}$. For the sake of contradiction, suppose $g_\alpha \neq 0$ for some α of degree $N > n$ maximal. Among those α, select those with α_1 maximal, among those consider the ones with α_2 maximal, etc. I.e. take α maximal for the lexicographic order. By (F.9).3, for every f,

$$[D^\alpha, f] = \sum_{j=1}^m \alpha_j \frac{\partial f}{\partial x^j} \prod_{k=1}^m \left(\frac{\partial}{\partial x^k}\right)^{\alpha_k - \delta_{kj}} + P(\partial/\partial x^i)$$

where P is a polynomial (depending on f) of degree $\leqslant N-2$. Take j maximal with $\alpha_j > 0$. Then

$$[D, f] = g_\alpha \alpha_j \frac{\partial f}{\partial x^j} \prod_{k=1}^j \left(\frac{\partial}{\partial x^k}\right)^{\alpha_k - \delta_{kj}} + Q(\partial/\partial x^i)$$

where Q is a linear combination of monomials D^β with $\beta < \alpha$ for the lexicographic order. By assumption, $[D, f]$ has degree $\leqslant n-1 < N-1$ so the first term should be zero. But if we take a point $p \in U$ for which $g_\alpha(p) \neq 0$ and choose f with $\partial f/\partial x^j(p) \neq 0$, we see that it is nonzero. Contradiction. ☐

Notation F.11. We denote the $C^\infty(M)$-module of differential operators on U of degree at most n by $\mathrm{E}_{\leqslant n}(M)$; their union as $\mathrm{E}_{<\infty}(M)$.

Remark F.12 (Differential operators and jet bundles)**.** For any vector bundle $E \to M$ we can define the kth jet bundle $J^k E$ which carries the information of partial derivatives op to order k of sections of E. It comes with a natural map $j^k : \Gamma(E) \to \Gamma(J^k E)$. We are interested in the case $E = M \times \mathbb{R}$, where $\Gamma(E) = C^\infty(M)$. For a differential operator D of degree $\leqslant k$ on M there exists a unique homomorphism of vector bundles $\widetilde{D} : J^k E \to E$ such that $D(f) = \widetilde{D} \circ j^k(f)$. That is, $\mathrm{E}(M)$ is linearly isomorphic with the space of sections of the Hom-bundle $\mathrm{Hom}(J^k E, E)$. Taking $D|_p$ corresponds to taking the fiber of $\widetilde{D}$ (as an section of the bundle).

Proposition F.13 (The filtered algebra of differential operators)**.** $\mathrm{E}_{<\infty}$ is filtered by the $\mathrm{E}_{\leqslant n}$.

Proof. From (F.9)(1). ☐

Proposition F.14 (Pushforward and filtration)**.** If $\sigma : M \to N$ is a diffeomorphism between manifolds, $\sigma_* : D_{<\infty}(M) \to D_{<\infty}(N)$ is an isomorphism of filtered algebras.

The key is that the grading is defined in charts, and composing a chart of N with σ yields a chart of M. That is, the hard work was to prove the equivalence of the definitions at (F.10).

Proof. Let (U, ϕ) be a chart of N, then $(\sigma^{-1}(U), \phi \circ \sigma)$ is a chart of M. Let $D \in \mathrm{E}_{\leqslant n}(M)$, then $D \in \mathrm{E}_{\leqslant n}(\sigma^{-1}(U))$ and hence $\phi_* \sigma_* D = (\phi \circ \sigma)_* D \in \mathrm{E}_{\leqslant n}(\phi(U))$ because $\phi \circ \sigma$ is a chart. That is, $\sigma_* D \in \mathrm{E}_{\leqslant n}(U)$. Because U is arbitrary, $\sigma_* D \in \mathrm{E}_{\leqslant n}(N)$. It remains to show that σ_* does not decrease degrees. This follows by symmetry, by applying what we just proved to σ_*^{-1}, which is the inverse of σ_*. ☐

F.1.2 Grading at a point

Definition F.15 (Degree of a differential operator at a point)**.** Let M be a manifold and $p \in M$. Then $P \in \mathrm{E}(M)_p$ has degree at most $\leqslant n$ if the following equivalent conditions hold:

1. For every chart around p, the image of P under the isomorphism $\mathrm{E}(M)_p \xrightarrow{\sim} \mathrm{Sym}(T_p M)$ from (F.4) has degree $\leqslant n$.

2. The above holds for one chart $(U, (x^i))$ around p.

3. It is the image of a differential operator of degree $\leqslant n$ under the map $\mathrm{E}(M) \to \mathrm{E}(M)_p : D \mapsto D|_p$.

Proof of equivalence. 1 $\implies$ 2: Ok. 2 $\implies$ 3: Because $P(\partial/\partial x^i)$ extends to U, and using a bump function smoothly to M. 3 $\implies$ 1: Ok. ☐

Proposition F.16 (Pushforward and filtration at a point)**.** If $\sigma : M \to N$ is a diffeomorphism between manifolds, $p \in M$ with $\sigma(p) = q$ and $D \in \mathrm{E}(M)$, then $D|_p$ and $(\sigma_* D)|_q$ have the same degree.

Proof. This is similar to the proof of (F.14). Let (U, ϕ) be a chart around q. Then $(\sigma^{-1}(U), \phi \circ \sigma)$ is a chart of M around p. By assumption, D has degree $\deg D$ at p in this chart, i.e. $(\phi \circ \sigma)_* D$ has degree $\deg D$ at $\phi(q)$. Thus $\sigma_* D$ has degree $\deg D$ at q. ☐

Again, the hard work has been done in the equivalence of definition of the global degree of a differential operator, where a crucial argument is the characterization of degrees with commutators. An alternative proof, which does not rely on that characterization, uses the following calculation:

Lemma F.17. Let $\sigma : U \to V$ be a diffeomorphism between open sets of $\mathbb{R}^n$, and $p \in U$ with $\sigma(p) = q$. (F.4) gives $\mathbb{R}$-linear isomorphisms $\mathrm{E}(M)_p \xrightarrow{\sim} \mathrm{Sym}(T_p M)$ and $\mathrm{E}(N)_q \xrightarrow{\sim} \mathrm{Sym}(T_q N)$. Let D^α be a monomial of degree a in the $\partial/\partial x^i$. Then $\mathrm{Sym}(d\sigma|_p) D|_p$ and $(\sigma_* D)|_q$ differ by a polynomial of degree $\leqslant a-1$ in the $\partial/\partial x^i|_q$.

Proof. By induction on a. For $a \leqslant 1$ we have $\mathrm{Sym}(d\sigma|_p) D|_p = (\sigma_* D)|_q$. Let $a > 1$ and write $D^\alpha = D^\gamma \partial/\partial x^i$. We have for $f \in C^\infty(N)$:

$$
\begin{aligned}
(\sigma_* D)|_q(f) &= \left(D^\gamma \frac{\partial}{\partial x^i} \right) (f \circ \sigma)(p) \\
&= D^\gamma|_p \left(\sum_j \frac{\partial f}{\partial x^j} \circ \sigma \cdot \frac{\partial \sigma_j}{\partial x^i} \right) \\
&= \sum_j D^\gamma|_p \left(\frac{\partial f}{\partial x^j} \circ \sigma \right) \cdot \frac{\partial \sigma_j}{\partial x^i}(p) + \sum_j P_j \left(\frac{\partial}{\partial x^k} \right)|_p \left(\frac{\partial f}{\partial x^j} \circ \sigma \right)
\end{aligned}
$$

with P_j of degree $\leqslant a-2$, by (F.9)(2).

$$
\begin{aligned}
&= \sum_j \mathrm{Sym}(d\sigma|_p) D^\gamma|_p \left(\frac{\partial f}{\partial x^j} \cdot \frac{\partial \sigma_j}{\partial x^i}(p) \right) \\
&\quad + \sum_j Q \left(\frac{\partial}{\partial x^k}|_q \right) \left(\frac{\partial f}{\partial x^j} \right) \cdot \frac{\partial \sigma_j}{\partial x^i}(p) \\
&\quad + P \left(\frac{\partial}{\partial x^k} \right)|_q
\end{aligned}
$$

with $\deg Q \leqslant a-2$ by the induction hypothesis, and $\deg P \leqslant a-1$. ☐

Second proof of (F.16). Choose charts (x^i) and (y^j) about p and q. (F.4) gives $\mathbb{R}$-linear isomorphisms $\mathrm{E}(M)_p \xrightarrow{\sim} \mathrm{Sym}(T_p M)$ and $\mathrm{E}(N)_q \xrightarrow{\sim} \mathrm{Sym}(T_q N)$. The idea is that taking σ_* is approximately the same as applying $\mathrm{Sym}(d\sigma|_p)$, which is a graded isomorphism. WLOG assume $a := \deg(D|_p) \leqslant \deg(\sigma_* D|_q) =: b$. We have $\sigma_* D|_q = \mathrm{Sym}(d\sigma|_p)(D|_p) + P(\partial/\partial y^j)$ with P of degree $\leqslant a-1 \leqslant b-1$ by (F.17). Thus

$$a = \deg(D|_p) = \deg(\mathrm{Sym}(d\sigma|_p)(D|_p)) = \deg\left(\sigma_* D|_q - P(\partial/\partial y^j)\right) = b$$ ☐

Proposition F.18 (Kernel of a differential operator)**.** Let D be a differential operator with real-valued coefficients on an open interval $I \subseteq \mathbb{R}$, of degree $n \geqslant 0$, and whose highest degree coefficient is nonzero on I. Then any local solution to $Df = 0$ extends uniquely to a global solution on I, and the kernel $\ker D$ has dimension exactly n. If D has complex-valued coefficients, the same holds, where the complex dimension equals n.

Proof. We can divide by the highest degree coefficient and write the equation $Df = 0$ in the form $\left(\frac{d}{dt}\right)^n f = F\left(\left(\frac{d}{dt}\right)^k f\right)_{k<n}$ for some smooth $F : \mathbb{R}^n \to \mathbb{R}$. By the uniqueness and existence theorem for linear ODE's, for any $t_0 \in I$ and any initial values $\left(\frac{d}{dt}\right)^k f(t_0)$ for $k \in \{0, \dots, n-1\}$ there is a unique local solution and it has a unique global extension. The claims follow.
For D complex, the same theorem says that the kernel has real dimension $2n$. It has indeed complex structure, and so it is of complex dimension n. □

F.1.3 Symbols

Proposition F.19. Let M be a manifold with a global chart (x^i) and $p \in M$. The $C^\infty(M)$-linear map

$$C^\infty(M) \otimes_{\mathbb{R}} \operatorname{Sym}(T_pM) \to \mathrm{E}_{<\infty}(M)$$

obtained by sending $\partial/\partial x^i|_p \mapsto \partial/\partial x^i$ is an isomorphism of graded $C^\infty(M)$-modules, where the LHS inherits the grading from $\operatorname{Sym}(T_pM)$.

Proof. In general, if V and W are real vector spaces and W has basis (e_i), then every element of $V \otimes_{\mathbb{R}} W$ can be written uniquely as a finite sum of $v \otimes e_i$ with $v \in V - \{0\}$. We conclude using (F.4). □

Definition F.20 (Symbols)**.** Fix a point $p \in M$, a global chart $\phi = (x^i)$ and use it to identify $\operatorname{Sym}(T_pM)$ with $\mathbb{R}[\xi_1, \dots, \xi_n]$ by sending $\partial/\partial x^i|_p \mapsto \xi_i$. The isomorphism of graded $C^\infty(M)$-modules

$$\mathrm{E}_{<\infty} \xrightarrow{\sim} C^\infty(M) \otimes_{\mathbb{R}} \operatorname{Sym}(T_pM) \xrightarrow{\sim} C^\infty(M) \otimes_{\mathbb{R}} \mathbb{R}[\xi_1, \dots, \xi_n]$$

is called the *(total) symbol map*. If $D \in \mathrm{E}_{<\infty}$ has degree d, the degree d component of the total symbol is called the *principal symbol* $\sigma(D)$.

The total symbol depends on the choice of the chart, but not on the point. The way in which it changes when taking a different chart $\psi = (y^i)$ is complicated and hard to understand. For the principal symbol however, we know by (F.17) that it transforms simply by the action of $\operatorname{Sym}(d(\psi \circ \phi^{-1}))$:

(F.21) $$\sigma_\psi(D) = \operatorname{Sym}(d(\psi \circ \phi^{-1}))\sigma_\phi(D)$$

F.2 Elliptic regularity

Definition F.22. Let M be a manifold with a global chart ϕ and $D \in \mathrm{E}_{<\infty}$ of degree d, with principal symbol $\sigma(D) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha(x)\xi^\alpha$. We call D:

1. elliptic if $\sigma(D)(x, \xi)$ is nonzero for all $x \in \phi(M)$ and $\xi \neq 0$.
2. uniformly elliptic if $\sigma(D)(x, \xi) \geqslant C\,\|\xi\|^d$ for some $C > 0$ independent of x.

Note that by (F.21), these conditions do not depend on the choice of a chart. Many theorems about existence and regularity of solutions to differential equations, are known under the name "elliptic regularity". We mention a few results that are relevant to us.

Theorem F.23 (Elliptic regularity for degree 2 operators)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open[18] and $D \in \mathrm{E}_{<\infty}(\Omega)$ of degree 2 uniformly elliptic and $f \in C^2(\Omega)$ with $Df = 0$.

1. If D has C^∞ coefficients (which we have always assumed) then $f \in C^\infty(\Omega)$.
2. If D has real analytic coefficients, then f is real analytic.

Proof. See e.g. [Evans, 2010, §6.3.1, Theorem 3] resp. [Petrowsky, 1939]. □

Proposition F.24. Let M be a Riemannian manifold and $-\Delta$ its Laplacian. Then Δ is locally uniformly elliptic of degree 2.

[18]Not necessarily bounded, contrary to what many texts assume.

Proof. By (D.45), we have, in a chart (x^i):

$$\Delta = \sum_{i,j} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right)$$

whose principal symbol in that chart is

$$\sum_{i,j} g^{ij} \xi_i \xi_j$$

That is, it is a bilinear form whose matrix is the inverse of that of the Riemannian metric. In particular, it is positive definite. □

Example F.25. For $M = \mathbf{H}^{n+1}$ with the half-space model, we have from (D.46):

$$-\Delta = \begin{cases} -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) & : n = 1 \\ -y^2 \left(\sum_i \frac{\partial^2}{\partial (x^i)^2} + \frac{\partial^2}{\partial y^2} \right) + (1-n) y \frac{\partial}{\partial y} & : n \geqslant 1 \end{cases}$$

Its coefficients are real analytic, hence elliptic regularity implies that C^2 Laplacian eigenfunctions are real analytic.

Theorem F.26 (Elliptic regularity for weak solutions)**.** [Agmon, 1965, Theorem 6.6] Let $\Omega \in \mathbb{R}^n$ be open and $D \in \mathrm{E}_{<\infty}(\Omega)$ be elliptic., $f \in C^\infty(\Omega)$ and $u \in L^2_{\mathrm{loc}}$ a locally square integrable weak solution to $Du = f$, in the sense that

$$\int_\Omega u D\phi = \int_\Omega f\phi$$

for all compactly supported test functions $\phi \in C_0^\infty(\Omega)$. Then $u \in C^\infty$, that is has a smooth representative. More generally, this holds if u is a simultaneous solution of an *overdetermined elliptic system* of differential operators D_i, meaning that their principal symbols do nowhere simultaneously vanish [Agmon, 1965, Definition 6.3].

F.3 Invariant differential operators

Definition F.27 (Invariant differential operator)**.** Let G be a Lie group and M a homogeneous G-space. An invariant differential operator on M is one for which, for all $f \in C^\infty(M)$ and $g \in G$:

$$D(f \circ g) = (Df) \circ g$$

That is, for all $x \in M$:

$$D|_x (f \circ g) = D|_{gx} f$$

or in terms of pullback: $D^g = D$.

This applies in particular to a homogeneous Riemannian manifold. A Lie group G acts on itself by left-translations $L_g : h \mapsto gh$ but also by right translations $R_g : h \mapsto hg$; this gives rise to the notions of left- and right-invariance.

Notation F.28. The algebra of invariant differential operators will be denoted $\mathscr{D}(M)$, the group G being implicit.

Remark F.29. Take a Lie group G with identity e, fix a metric on $T_e G$ and transport it to obtain a Riemannian metric that is invariant by G. Then G is contained in the isometry group, but usually not equal, for example for $G = \mathbb{R}^n$. We see that differential operators that are invariant by G, need not be invariant by its isometry group.

Proposition F.30. Take a homogeneous G-space M and $p \in M$. A left-invariant extension D of a differential operator $D|_p$ at a point, if it exists, is unique.

Proof. Because $D|_{gp}(f) = D|_p(f \circ g)$ and G acts transitively. □

Proposition F.31. For $D \in \mathscr{D}(M)$ and $n \in \mathbb{N}$, TFAE:

1. $D \in \mathrm{E}_{\leqslant n}$.
2. There exists a point $p \in M$ with $D|_p \in \mathrm{Sym}(T_pM)_{\leqslant n}$.

Proof. From (F.16). □

Example F.32. Let M be a Riemannian manifold with isometry group G. The Laplacian is G-invariant of degree 2.

Proof. It is a differential operator of degree at most 2 by the explicit formula from (D.45). It has degree exactly 2 by (F.9)(2). The invariance follows from (F.33) below. □

Proposition F.33. Let M, N be Riemannian manifolds and $\phi : M \to N$ an isometry. Then for $f \in C^\infty(N)$ and $X \in \Gamma(M)$:

1. $d\phi \operatorname{grad}(f \circ \phi) = \operatorname{grad} f$.
2. $\operatorname{div}(d\phi X) \circ \phi = \operatorname{div} X$.
3. $-\Delta(f \circ \phi) = -\Delta(f) \circ \phi$.

Proof. 1. For a vector field $X \in \Gamma(M)$ we have

$$\begin{aligned}\langle \operatorname{grad}(f \circ \phi), X\rangle &= d(f \circ \phi)X \\ &= df\, d\phi X \\ &= \langle \operatorname{grad} f, d\phi X\rangle \\ &= \left\langle (d\phi)^{-1} \operatorname{grad} f, X\right\rangle\end{aligned}$$

2. Let α resp. β be the Riemannian volume form of M resp. N. We have

$$\phi^*\left(i_{d\phi X}\beta\right) = i_X\left(\phi^*\beta\right)$$

and $\phi^*\beta = \alpha$. The exterior derivative commutes with ϕ^*, so

$$\phi^*\left(\operatorname{div}(d\phi X)\beta\right) = \operatorname{div}(X)\alpha$$

and we conclude using $\phi^*\beta = \alpha$ once more.

3. From the first two formulas applied to ϕ and ϕ^{-1}:

$$\operatorname{div}\operatorname{grad}(f \circ \phi) = \operatorname{div}\left((d\phi)^{-1}\operatorname{grad} f\right) = (\operatorname{div}\operatorname{grad} f) \circ \phi$$ □

F.4 Differential operators on Lie groups

Proposition F.34 (Invariant vector fields)**.** Let G be a Lie group with identity e. For every tangent vector $X \in T_eG = \mathfrak{g}$, there is a unique left-invariant smooth extension to a vector field on G by defining $\widetilde{X}_g = dL_g(X_e)$.

Proof. Uniqueness follows from (F.30). For existence, we show that the construction in the statement is indeed smooth. For $X \in \mathfrak{g}$ we have $X(f) = \frac{d}{dt} f(\exp(tX))|_{t=0}$. Hence $\widetilde{X}_g(f) := \frac{d}{dt} f\left(g \exp(tX)\right)|_{t=0}$ is smooth in g. It is a left-invariant extension by construction. □

Taking the unique left-invariant extension $\widetilde{X}$ of a tangent vector $X \in \mathfrak{g}$ gives an $\mathbb{R}$-linear map $\lambda : \mathfrak{g} \to \mathscr{D}(G)$.

Lemma F.35. For:

1. a real function f that is smooth in a neighborhood of $t_0 \in \mathbb{R}$:

$$\frac{d}{dt_1} \cdots \frac{d}{dt_n} f(t_1 + \cdots + t_n)|_{t_i = t_0} = \frac{d^n}{dt^n} f(t)|_{t = nt_0}$$

2. a function $f : \mathbb{R}^n \to \mathbb{R}$ smooth in a neighborhood of t_0 and $x_1, \ldots, x_n \in \mathbb{R}$:

$$\frac{d^k}{dt^k} f(x_1 t, \ldots, x_n t)|_{t=t_0} = \left(\sum x^i \frac{\partial}{\partial t_i} \right)^k f(x_1 t_0, \ldots, x_n t_0)$$

Proof. 1. By induction: write the innermost derivative as

$$\frac{d}{dt_n} f(t_n)|_{t_n = t_0 + t_1 + \cdots + t_{n-1}}$$

and apply the induction hypothesis to $\frac{d}{dt} f(t_0 + t)$.

2. Again by induction. The case $n = 1$ is the chain rule, and we apply the induction hypothesis to $\frac{d}{dt} f(x_1 t, \ldots, x_n t)$. □

Proposition F.36 (The symmetrization map)**.** Let $n = \dim(G)$. There exists a unique extension of λ to a linear map $\lambda : \mathrm{Sym}(\mathfrak{g}) \to \mathscr{D}(G)$ satisfying $\lambda(X^m) = \widetilde{X}^m$ for $X \in \mathfrak{g}$, called *symmetrization.* Moreover, it is a filtered linear isomorphism that respects degrees, and we have the following explicit formula: Let (X_i) be a basis of $\mathfrak{g}$. Then $\phi : (t_i) \mapsto \exp\left(\sum_i t_i X_i\right)$ is a local parametrization of G around e. Denote $\partial_i = \partial/\partial t_i$ so that $\partial_i|_e = X_i$. Then for a real polynomial P in n variables and $f \in C^\infty(G)$,

$$\lambda(P(X_i))|_g(f) = P(\partial_i) f\left(g \exp\left(t_1 X_1 + \cdots + t_n X_n\right)\right)|_{t_i = 0}$$

Proof. [Helgason, 1984, Theorem II.4.3] It is a general fact that the symmetric algebra $\mathrm{Sym}(V)$ of a vector space over a field of characteristic 0 is linearly generated by the powers of elements of V. (More precisely, the set of elements of degree n is linearly generated by nth powers.) Thus there is at most one such linear map.

The definition of $\lambda(P(X_i))$ in the statement sends a smooth f to a smooth function. It is a differential operator because it is linear and decreases supports. It is left-invariant, essentially by construction. The map λ is linear. For each i we have $\lambda(X_i)_g f = \partial_i|_e f(g \exp(t_i X_i) = \widetilde{X}_i|_g(f)$ by definition of exp and $\widetilde{X}_i$. By linearity, $\lambda(X) = \widetilde{X}$ for all $X \in \mathfrak{g}$, so we have an extension of the map λ we defined earlier. The map λ is surjective because it gives a left-invariant extension of any $Q \in \mathrm{Sym}(\mathfrak{g})$, which is unique. For injectivity, take a monomial in the X_i of maximal degree whose coefficient at e is nonzero. Choose f such that $f\left(\exp\left(t_1 X_1 + \cdots + t_n X_n\right)\right)$ equals that monomial in the t_i in a neighborhood of e. Then $\lambda(P)|_e(f) \neq 0$.

Consider now $\lambda(X^k)$ for $X \in \mathfrak{g}$ with $X = \sum x^i X_i$. Using (F.35) we have:

$$\begin{aligned}
\lambda(X^k)|_g(f) &= \left(\sum x^i \partial_i\right)^k f\left(g \exp\left(\sum t_i X_i\right)\right)|_{t_i=0} \\
&= \frac{d^k}{dt^k} f\left(g \exp\left(\sum t x^i X_i\right)\right)|_{t=0} \\
&= \frac{d^k}{dt^k} f\left(g \exp(tX)\right)|_{t=0} \\
&= \frac{d}{dt_1} \cdots \frac{d}{dt_k} f\left(g \exp\left(t_1 X + \cdots + t_k X\right)\right) \\
&= \frac{d}{dt_1} \cdots \frac{d}{dt_k} f\left(g \exp(t_1 X) \cdots \exp(t_k X)\right) \\
&= \widetilde{X}^k|_g(f)
\end{aligned}$$

Regarding degrees, we have that $\lambda(P)|_e = P$ so P and $\lambda(P)$ have the same degree by (F.31). □

Remark F.37. The symmetrization map need not be an algebra isomorphism.

Corollary F.38. The $\mathbb{R}$-vector space $\mathscr{D}(G)_{\leqslant k}$ has dimension $\binom{k+n}{n}$.

Proof. Because the symmetrization map is a filtered linear isomorphism that respects degrees. □

It is natural to consider the induced algebra homomorphism $T(\mathfrak{g}) \to \mathscr{D}(G)$ from the tensor algebra. The kernel contains the elements of the form $XY - YX - [X, Y]$. Call I the two-sided ideal generated by these elements. Then $U(\mathfrak{g}) = T(\mathfrak{g})/I$ is the universal enveloping algebra of $\mathfrak{g}$.

Proposition F.39. The filtered algebra homomorphism $U(\mathfrak{g}) \to \mathscr{D}(G)$ is an isomorphism.

Proof. It is definitely surjective. More precisely, the image of $T(\mathfrak{g})_{\leqslant k}$ in the quotient $U(\mathfrak{g})$ surjects onto $\mathscr{D}(G)_{\leqslant k}$. Hence it suffices to find a generating set of $T(\mathfrak{g})_{\leqslant k}/I$ with $\binom{k+n}{n}$ elements, and it will be automatically a basis. This is the easy part of Poincaré–Birkhoff–Witt! See below. □

Theorem F.40 (Poincaré–Birkhoff–Witt)**.** Let G be a real Lie group of dimension n with Lie algebra $\mathfrak{g}$. Let (e_i) be a basis of $\mathfrak{g}$.

1. The products $\prod e_{i_j}$ of length $m \leqslant k$ with $i_1 \leqslant \ldots \leqslant i_m$ generate $T(\mathfrak{g})_{\leqslant k}/I$ linearly over $\mathbb{R}$.
2. They form a basis of $T(\mathfrak{g})_{\leqslant k}/I$.

Proof.
1. We represent every element $Y \in T(\mathfrak{g})_{\leqslant k}/I$ as a linear combination, by induction on the degree of the highest occuring monomials in Y and the amount of them.
2. There are $\binom{k+n}{n}$ of them. They are linearly independent because $T(\mathfrak{g})_{\leqslant k}/I$ surjects onto $\mathscr{D}(G)_{\leqslant k}$, which has dimension $\binom{n+k}{n}$ by (F.38). □

G Spectral theory of the Laplacian

When studying differential operators and their spectral properties, the theory of bounded operators is rarely applicable or satisfactory: Given a Riemannian manifold M and a differential operator $D \in \mathrm{E}(M)$, one would like to study the action of D on $C^\infty(M)$, or at least on square-integrable functions: $C^\infty(M) \cap L^2(M)$. This is no longer a Banach space, and there is no reason to assume that D applied to a smooth square-integrable function yields again a square-integrable function. This leads to the notion of unbounded operators. We will give special attention to the Laplacian.

G.1 Unbounded operators

Definition G.1 (Unbounded operator)**.** An *unbounded operator* between Banach spaces X and Y is a linear map A from a subspace (not necessarily closed) $D \subseteq X$ to Y. We do not require A to be continuous on D. We call D the domain of A.

In this subsection, "operator" always means "unbounded operator". If D is dense in X, we say A is *densely defined*.

Example G.2. The space $C([0,1],\mathbb{R})$ of continuous $\mathbb{R}$-valued functions on $[0,1]$ is a Banach space for the supremum norm $\|\cdot\|_\infty$. It contains the continuously differentiable functions as a subspace. (But not a closed subspace: by stone Weierstrass already the polynomials are dense.) The differentiation operator defined on this subspace is unbounded, because there exist uniformly bounded functions with arbitrary steep slope at some point.

Definition G.3 (Extensions and closure)**.** Let X be a Hilbert space. When A and B are operators on X, B is defined on the domain of A and coincides with A on its domain, we call B an *extension* of A, denoted $A \subseteq B$. We say A is *closed* when its graph

$$\operatorname{gra} A \subseteq X \times X$$

is closed. Otherwise, consider the closure $\overline{\operatorname{gra} A}$. When this is the graph of an operator B, we call A *closable* and B its *closure*.

By the closed graph theorem (A.14), bounded operators are always closed. Contrary to what the terminology suggests, the closure (when it exists) is in general not defined on the closure of the original domain. Otherwise, together with the closed graph theorem that would imply that every closable densely-defined operator is bounded.

Definition G.4 (Symmetric operator)**.** Let X be a Hilbert space and A a densely defined operator with domain D. We say A is *symmetric* if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \qquad \forall x, y \in D$$

Proposition G.5. A symmetric operator on a Hilbert space is closable, and its closure is symmetric.

Proof. See [Davies, 1995, Lemma 1.1.4]. □

Definition G.6. A densely defined operator A with domain D on a complex Hilbert space is *positive* iff its associated quadratic form is nonnegative:

$$\langle Ax, x\rangle \geqslant 0 \qquad \forall x \in D$$

Proposition G.7. Let A be a densely defined symmetric positive operator on a complex Hilbert space H. Then its closure is positive.

Proof. Let D be the domain of A, then by definition, the domain of its closure $\overline{A}$ is the set of $f \in H$ for which there exists $g \in H$ and a sequence $(f_n) \in D$ such that $f_n \to f$ and $Af_n \to g$, in which case $\overline{A}f = g$. Then, by continuity of the inner product, we have

$$\langle \overline{A}f, f\rangle = \lim_{n\to\infty} \langle Af_n, f_n\rangle \in \mathbb{R}_{\geqslant 0}$$ □

It is tempting to refer to symmetric operators as self-adjoint ones. Indeed, in the case of bounded operators, these notions coincide:

Definition G.8 (Adjoint of an unbounded operator)**.** Let X be a Hilbert space and A a densely defined operator with domain D. Its adjoint is the operator A^* with domain

$$D' = \{y \in X : \exists z \in X : \forall x \in X : \langle Ax, y\rangle = \langle x, z\rangle\}$$

and one defines $A^* y = z$ for such y. We say A is self-adjoint if $D = D^*$ and $A = A^*$.

Note that A^* is well-defined because A is assumed densely defined: if

$$\langle x, z_1 - z_2\rangle = 0 \qquad \forall x \in D$$

then $z_1 = z_2$. For a symmetric operator, we always have $A \subseteq A^*$. A self-adjoint operator is always symmetric, but the converse is not true.

Proposition G.9. If A is closed and densely defined, then A^* is also closed and densely defined.

Proof. See [Davies, 1995, Lemma 1.2.1]. □

Definition G.10. Let X be a complex Hilbert space and A a densely defined operator on X. We call A *essentially self-adjoint* if it satisfies the following equivalent conditions:

1. A is symmetric and its closure is self-adjoint.
2. A has a unique self-adjoint extension.

Proof of equivalence. See [Davies, 1995, Theorem 1.2.7]. □

Proposition G.11. Let $A : X \to Y$ be an everywhere defined operator between Banach spaces. If it has a bounded inverse, then $\|Ax\| \gg \|x\|$ uniformly in x.

Proof. We have, for $x \in X$,

$$\|x\| = \left\|A^{-1}Ax\right\| \leqslant \left\|A^{-1}\right\| \|Ax\|$$ □

G.2 The Laplacian as a symmetric operator

Let M be a Riemannian manifold. We want to understand the spectrum of the Laplacian $-\Delta$. For the moment, $-\Delta$ is defined on $C^\infty(M)$, and the subspace $C_0^\infty(M)$ of compactly supported smooth functions is stable. We have an inclusion

$$C_0^\infty(M) \subseteq L^2(M)$$

and one can wonder what the theory of extensions of symmetric unbounded operators tells us about the action of $-\Delta$ on $L^2(M)$, or at least, $L^2(M) \cap C^\infty(M)$.
Note that the Laplacian of a smooth L^2 function need not be L^2 again, take for example $x^{-1/3}$ on $]0, 1]$ with the standard Euclidean metric.

Definition G.12. Let M be an oriented Riemannian manifold with smooth boundary ∂M, given the induced orientation. The *outward unit normal* is the unique section $N : \partial M \to TM$ for which if $p \in \partial M$ and (e_i) is a positive orthonormal basis for $T_p\partial M$ then $(N, (e_i))$ is a positive orthonormal basis of T_pM.

The boundary ∂M is a Riemannian manifold for the induced metric; we denote its volume form by $d\widetilde{V}$.
is $i_N(dV)|_{\partial M}$, where dV is the volume form of M. (Since both agree on an orthonormal basis of $T_p\partial M$.)

Proposition G.13. Let M be a Riemannian manifold with volume form dV and smooth boundary $\partial M \overset{j}{\hookrightarrow} M$. Let X be a vector field on M and u smooth on M. Then

1. $d\widetilde{V} = j^*(i_N(dV))$.

2. $j^*(i_X dV) = \langle X, N \rangle \, d\widetilde{V}$.

3. $\operatorname{div}(uX) = u \cdot \operatorname{div} X + \langle \operatorname{grad} u, X \rangle$.

Proof. 1. Since both agree on an orthonormal basis of $T_p\partial M$, for each $p \in \partial M$.

2. The same argument: take a local orthonormal frame (E_i) of $T\partial M$ and write $X = \langle X, N \rangle N + \sum X_i E_i$. Evaluating both sides on the frame (E_i) gives the same result.

3. We have $\operatorname{div}(uX)dV = d(i_{uX}dV) = d(u \cdot i_X dV) = ud(i_X dV) + du \wedge i_X dV$. The first term is $u \operatorname{div} X dV$; the second is $i_X(du) \wedge dV = X(u)dV$ where we use the fact that $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge i_X\beta$.

□

Theorem G.14 (Stokes)**.** Let ω be a compactly supported $(n-1)$-form on an oriented manifold M of dimension n with (or without) smooth boundary. Then

$$\int_M d\omega = \int_{\partial M} \omega$$

for the induced orientation on ∂M. In particular if M has no boundary, then the LHS is 0.

Proposition G.15 (Green's identities)**.** Let M be an oriented Riemannian manifold with measure dV, smooth (possibly empty) boundary ∂M with measure $d\widetilde{V}$ and u, v smooth functions, at least one of which has compact support. Then

1.
$$\int_M u(-\Delta v)dV - \int_M \langle \operatorname{grad} u, \operatorname{grad} v \rangle \, dV = -\int_{\partial M} uNv d\widetilde{V}$$

2.
$$\int_M (u(-\Delta v) - (-\Delta u)v)dV = -\int_{\partial M} (uNv - vNu)d\widetilde{V}$$

where N denotes the outward unit normal vector field.

Proof. 1. From Stokes and (G.13). 2. Immediate from 1. □

Definition G.16. A *harmonic function* is one whose Laplacian is 0.

There are a few things that one can deduce from Green's identities. We mention the ones that are relevant to us:

Corollary G.17. Let M be an oriented Riemannian manifold without boundary.

1. 0 is an eigenvalue of $-\Delta$.

2. $-\Delta$ is symmetric for the L^2-inner product: if $u, v \in C_0^\infty(M, \mathbb{C})$ have compact support, then
$$\int_M u \cdot \overline{-\Delta v} dV = \int_M -\Delta u \cdot \bar{v} dV$$

3. $-\Delta$ is a positive operator on compactly supported functions: if $u \in C_0^\infty(M, \mathbb{C})$ has compact support, then
$$\int_M u \cdot \overline{-\Delta u} dV \in \mathbb{R}_{\geqslant 0}$$
In particular, its eigenvalues are nonnegative.

Proof. 1. A nonzero constant function is harmonic.

2. For real-valued u and v this follows directly from Green's identities. For $u = u_1 + iu_2$ and $v = v_1 + iv_2$ complex-valued, this follows by taking linear combinations, as in the lemma below.

3. For real-valued functions this follows by the first of Green's identities. For complex-valued functions we conclude again by taking linear combinations, or using the general lemma below. □

Lemma G.18. Let X be a Hilbert space over $\mathbb{R}$ and A a densely defined symmetric unbounded operator on X with domain D, that is,

$$\langle Au, v\rangle = \langle u, Av\rangle \qquad \forall u, v \in D$$

Let $X_{\mathbb{C}}$ be its complexification and $A_{\mathbb{C}}$ the induced operator on $X_{\mathbb{C}}$ with domain $D_{\mathbb{C}}$, so that $A_{\mathbb{C}}(u_1 + iu_2) = Au_1 + iAu_2$. Then:

1. $A_{\mathbb{C}}$ is symmetric.

2. If A is positive in the sense that $\langle Au, u\rangle \in \mathbb{R}_{\geqslant 0}$ for all $u \in D$, then so is $A_{\mathbb{C}}$.

Proof. 1. From the definition of $A_{\mathbb{C}}$ and the properties of the complexified inner product.

2. Because A is assumed symmetric:

$$\begin{aligned}\langle A(u + iv), u + iv\rangle &= \langle Au, u\rangle + \langle Av, v\rangle + \langle Au, iv\rangle + \langle iAv, u\rangle \\ &= \langle Au, u\rangle + \langle Av, v\rangle \geqslant 0\end{aligned}$$

□

G.3 Extensions and essential self-adjointness

We have established that Laplacian is a positive symmetric unbounded operator on $L^2(M)$, whose domain contains the compactly supported smooth functions. It is densely defined:

Proposition G.19. Let M be an orientable Riemannian manifold, and write it as the countable increasing union of compact subsets K_n. Let $f \in L^2(M)$. Let $\delta_n : M \to [0, 1]$ be compactly supported smooth functions which are 1 on K_n. Then

$$\lim_{n\to\infty} \int_M |f - f\delta_n|^2 = 0$$

Proof. Follows immediately from the dominated convergence theorem. □

Note that if M is complete, it is σ-compact: Fix $x_0 \in M$, then the closed geodesic balls $\overline{B}(x_0, n)$ are compact by Hopf–Rinow (D.39), and they cover M.

Theorem G.20. Let M be a complete Riemannian manifold. Then the Laplacian, as a densely defined unbounded operator on $L^2(M)$ with domain $C_0^\infty(M)$, is essentially self-adjoint.

Proof. This was first shown by Gaffney in [Gaffney, 1951] and Roelcke in [Roelcke, 1960]. A different proof is given by Strichartz in [Strichartz, 1983]. A discussion of essential self-adjointness of the Laplacian can also be found in the blog post [Tao, 2011]. □

Assume that M is complete from now on, which is in particular the case if M is homogeneous (E.4), for example when $M = \mathbf{H}$ is the hyperbolic plane, and we observed that this implies that finite-volume quotients $\Gamma\backslash\mathbf{H}$ are also complete. The Laplacian is then essentially self-adjoint, so it has a unique self-adjoint extension, which is closed: it is the closure of the Laplacian on compactly supported smooth functions. It is called the *Dirichlet–Laplacian.* We will keep calling it simply the Laplacian and denote it by $-\Delta$.

Proposition G.21. The domain of $-\Delta$ contains

$$\{f \in C^\infty(M) : f \in L^2(M), -\Delta f \in L^2(M)\}$$

Proof. Follows from the discussion in [Strichartz, 1983]. □

One can show that the domain of $-\Delta$ is exactly equal to the Sobolev space H^2, and that on smooth L^2-functions, the closure of the Laplacian coincides with the usual Laplacian. Finally, we conclude, using (G.7) on the closure of a positive operator:

Proposition G.22. The Laplacian is a positive densely-defined operator on $L^2(M)$. In particular, its eigenvalues are nonnegative.

G.4 Operators with compact resolvent

We call an unbounded operator $T : X \to Y$ between normed spaces *bounded below* if there exists $C > 0$ for which

$$\|Tx\| \geqslant C\,\|x\| \qquad \forall x \in X$$

By Hahn-Banach, this is equivalent to T having a bounded left inverse. In particular, a bounded below operator is injective. A closed bounded below unbounded operator between Banach spaces has closed range.

Definition G.23. Let H be a complex Hilbert space and T a densely defined operator on H with domain D.

1. The *resolvent set* $\rho(T)$ is the set of all $\lambda \in \mathbb{C}$ for which $\lambda - T$
   (a) is injective
   (b) has dense range $\mathrm{ran}(\lambda - T)$
   (c) its inverse $(\lambda - T)^{-1} : \mathrm{ran}(\lambda - T) \to D$ is bounded.
2. The *spectrum* $\sigma(T) = \mathbb{C} - \rho(T)$ is its complement.

We say $\lambda \in \sigma(T)$ belongs to

- the *point spectrum* $\sigma_p(T)$ if $\lambda - T$ is not injective, that is, if λ is an eigenvalue.
- the *continuous spectrum* $\sigma_c(T)$ if $\lambda - T$ is injective with dense range but not surjective.
- the residual spectrum $\sigma_r(T)$ if $\lambda - T$ is injective and its range is not dense.

For $\lambda \in \rho(T)$, we call $R_\lambda = (\lambda - T)^{-1}$ the resolvent at λ. It commutes with T for all such λ.

Definition G.24 (Compact resolvent)**.** A densely defined operator T on H with domain D has *compact resolvent* if R_λ is a compact operator $H \to D$ for some $\lambda \in \rho(T)$.

Note that such T cannot be bounded on D, unless H is finite-dimensional: when T is bounded, the compact bounded operator $(\lambda - T)^{-1} : H \to D \subset H$ has a bounded left inverse (an extension of $\lambda - T$ from D to H, given by Hahn-Banach). But then 1_D is compact, which implies $\dim D = \dim H < \infty$. When T has compact resolvent, its spectrum inherits many properties from compact operators:

Proposition G.25. Let T be densely defined and closed on H with compact resolvent. Then the spectrum $\sigma(T)$

1. is discrete (hence countable)
2. has no accumulation point
3. consists of eigenvalues only, that is, equals the point spectrum σ_p.
4. The eigenspaces of T are finite-dimensional.

If T is in addition self-adjoint, then

5. H has an orthonormal basis of eigenvectors of T. In particular, H is separable.

Proof. [Taylor, 1997, Proposition 8.8]

1,2 Let $\zeta \neq 0$ be such that $\zeta - T$ has a compact inverse R_ζ. We know from (A.37) that $\sigma(R_\zeta)$ is countable and can only have 0 as an accumulation point. A computation shows that for $\lambda \in \rho(R_\zeta) - \{0\}$ we have $\zeta - \lambda^{-1} \in \rho(T)$ with

$$(\zeta - \lambda^{-1} - T)^{-1} = \lambda(\lambda - R_\zeta)^{-1} R_\zeta \tag{G.26}$$

The first two statements follow.

3 In fact, the inclusion

$$\{\zeta\} \cup \left\{\zeta - \frac{1}{\rho(R_\zeta)}\right\} \subseteq \rho(T) \tag{G.27}$$

is an equality: when $\lambda \in \sigma(R_\zeta)$ is nonzero, then it is an eigenvalue of $(\zeta - T)^{-1}$, hence $\zeta - \lambda^{-1}$ is an eigenvalue of T. In particular, $\zeta - \lambda^{-1} \in \sigma_p(T)$. Note how from (G.26) and (G.27) it now follows that all resolvents of T are compact.

4 The finite-dimensionality follows from the same argument: for $\mu \neq \zeta$, the μ-eigenspace of T is the $(\zeta - \mu)^{-1}$-eigenspace of R_ζ, which is finite-dimensional. For $\mu = \zeta$, it suffices to repeat the argument with another $\zeta \in \rho(T)$.

5 We know from the spectral theorem (A.49) that H has an orthonormal basis of eigenvectors of R_ζ, hence of T. Because the eigenspaces are finite-dimensional, H must be separable. □

Conversely, if H has an orthonormal basis of eigenvectors for T, and the eigenvalues of T are nonzero and tend to ∞ with multiplicities, then T has compact resolvent. Indeed, for λ not an eigenvalue, the resolvent R_λ exists, is diagonalizable and its eigenvalues tend to 0 with multiplicities. By (A.42), it follows that R_λ is compact.

G.5 The spectrum of the Laplacian

One can show that:

Theorem G.28. If M is a compact Riemannian manifold, then the resolvent of the (closure of the) Laplacian is compact. In particular:

1. The spectrum of $-\Delta$ equals its point spectrum, which is a discrete closed infinite subset of $\mathbb{R}_{\geqslant 0}$.
2. Each eigenspace is finite-dimensional and the eigenspaces corresponding to distinct eigenvalues are L^2-orthogonal.

Sketch of proof. There are two possible arguments. One is by explicitly writing the resolvent of the Laplacian as an integral operator in terms of what is called a *Green function*. See e.g. [Chavel, 1984, VI§1]. A different proof uses a general result in functional analysis to reduce the compactness of the resolvent to the Rellich–Kondrachov embedding theorem, which says that the embedding of the Sobolev space H^2 in L^2 is compact. See e.g. [Taylor, 2018, Proposition 2.8], where it is also shown that $[1, \infty] \subset \rho(-\Delta)$.
Once we know that $-\Delta$ has compact resolvent, the remaining statements follow from (G.25) and the positivity of the Laplacian. □

While (G.28) is no longer true for noncompact manifolds, the same techniques generalize to certain noncompact manifolds:

Theorem G.29. [Bump, 1996, Theorem 2.3.5] Let Γ be a lattice in $\mathsf{PSL}_2(\mathbb{R})$. Even when the quotient $\Gamma\backslash\mathbf{H}$ is not compact, $-\Delta$ has countably many eigenvalues, and they tend to infinity, counting multiplicities. In particular, its eigenspaces are finite-dimensional.

H Whittaker functions

We study the differential equation

$$G'' + (\lambda y^{-2} - 1)G = 0$$

when $\lambda \in \mathbb{C}$, which occurred naturally in the Fourier expansion of periodic Laplacian eigenfunctions on $\mathbf{H}$ (2.27). Substituting $G(y) = W(2y)$ gives

$$W''(2y) + \left(\frac{\lambda}{(2y)^2} - \frac{1}{4}\right) W(2y) = 0$$

This equation in W was proposed by Whittaker in the form

$$W'' + \left(\frac{\frac{1}{4} - m^2}{y^2} - \frac{1}{4}\right) W = 0 \tag{H.1}$$

Note that when $W(y)$ is a solution on $\mathbb{R} - \{0\}$, then so is $W(-y)$.

Proposition H.2. A solution of (H.1) is given by the Whittaker function[19] $W_{0,m}(y)$, which is analytic for $y \in \mathbb{C} - \mathbb{R}_{\leqslant 0}$ and for $\Re\mathrm{e}\left(m + \frac{1}{2}\right) > 0$ it equals

$$W_{0,m}(y) = \frac{e^{-y/2}}{\Gamma\left(\frac{1}{2} + m\right)} \int_0^\infty t^{m-1/2} \left(1 + \frac{t}{y}\right)^{m-1/2} e^{-t} dt$$

Where for $\log\left(1 + \frac{t}{y}\right)$ we take the principal branch, with branch cut $\mathbb{R}_{\leqslant 0}$.

Proof. See [Whittaker and Watson, 1915, Chapter XVI §16·12]. □

Thus the Witthaker function is analytic with the exception of a branch. We can turn its branch a little bit counterclockwise, say to $z_0\mathbb{R}_{\leqslant 0}$ for some z_0 of modulus 1 with $0 < \arg z_0 < \frac{\pi}{2}$, by shifting the contour to the half-line $z_0^{-1} \cdot \mathbb{R}_{\geqslant 0}$. (The exponential decay of the integrand ensures that we don't change the Witthaker function.) To do this, a branch must then be chosen for $\log\left(1 + \frac{tz_0}{y}\right)$. The principal branch of the logarithm makes the Witthaker function analytic with branch $z_0\mathbb{R}_{\leqslant 0}$, and we have only changed its values for $-\pi \leqslant \arg y \leqslant -\pi + \arg z_0$. In particular, and most importantly, we haven't changed the branch in the angular region $-\arg z_0 \leqslant \arg y \leqslant 0$, where the shifting of the contour happens.
Similarly, we can turn the branch clockwise.

H.1 Asymptotic expansion

Proposition H.3. The Whittaker function $W_{0,m}$, $\Re\mathrm{e}\left(m + \frac{1}{2}\right) > 0$ satisfies

$$W_{0,m}(z) = e^{-z/2}\left(1 + O_\alpha(z^{-1})\right) \tag{H.4}$$

as $|z| \to \infty$ in the angular region $|\arg z| \leqslant \pi - \alpha < \pi$.

Proof. See [Whittaker and Watson, 1915, Chapter XVI §16·3], where one can also find the full series expansion for the $O(z^{-1})$ term. □

Similarly, one proves that this asymptotic relation still holds when we turn the branch of $W_{0,m}$ a little bit clockwise or counterclockwise.

[19] One of the parameters in Whittaker's original equation is 0 in our case.

H.2 Second solution

When $W_{0,m}(y)$ is a solution to (H.1) and it is holomorphic on a neighborhood of the half-line $\mathbb{R}_{<0}$, then so is $W_{0,m}(-y)$. Moreover, the asymptotic expansion (H.4) and the discussion about shifting the branch cut shows that, for $\Re\mathrm{e}\left(m+\frac{1}{2}\right)>0$,

$$W_{0,m}(-z) = e^{z/2}\left(1+O_\alpha(z^{-1})\right) \tag{H.5}$$

for z in an angular region containing the positive real line.
From the asymptotic expansions we conclude in particular that $W_{0,m}(z)$ and $W_{0,m}(-z)$ are linearly independent. We conclude:

Proposition H.6. The equation (H.1) has two linearly independent solutions on $\mathbb{R}_{>0}$, $W_{0,m}(y)$ and $W_{0,m}(-y)$, the first decays exponentially, the other grows exponentially.

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