Errata master's thesis

Bart Michels

October 27, 2018

p. 8 [...] holomorphic on $U \subseteq \mathbb{C}$ with values in [...] That is, that the limit [...] exists for all $s_0 \in U$.

p. 8 In the theorem: the statement is not wrong, but $E(w, s) = \phi(s)E(w, 1-s)$ is in line with the definition of ϕ in Theorem 4.10.

p. 21 Remark 3.13: $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

p. 41 Item 3.: L^3 should be L_3.

p. 49 Below (5.3): it is k which is supported on point pairs at distance at most R, not K. It is $(k \star f)(z)$ which depends only on values of $f(w)$ with $d(z, w) < R$.

p. 49 Second equation: the first and second convolutions should be on the standard fundamental domain $\mathscr{F}$, but even then; the last equality is not obvious, because y^s is not Γ-invariant and (4.20) does not apply. We are here implicitly using a 'local' version of the Selberg eigenfunction principle: say k is supported on point pairs at distance $\leqslant R$, so that $(k \star f)(z)$ depends only on values of $f(w)$ for $d(w, z) < R$. Now suppose f is a Laplacian eigenfunction on $B(z, R)$ with eigenvalue $s(1-s)$. (Think $f = [\alpha(y)y^s]_{\mathscr{F}}$.) Under this weaker assumption, does the Selberg eigenfunction principle still hold at z, in the sense that $(k \star f)(z) = \widehat{k}(s)f(z)$? The answer is yes; the proof of the eigenfunction principle (3.31) carries through: if f is a Laplacian eigenfunction in a geodesic ball $B(z, R)$, then so is its radial symmetrization about z in that ball. Because k is supported on point pairs at distance less than R, we conclude using

$$(k \star f)^{\text{rad}}(z) = (k \star f_z^{\text{rad}})(z) = (k \star f(z) \cdot \omega_s(\cdot, z))(z) = f(z) \cdot \widehat{k}(s)$$

as in the proof of (3.31). In the case of $f = [\alpha(y)y^s]_{\mathscr{F}}$, which is a Laplacian eigenfunction for y large, note that we can take the R in the requirement for the support of k independent of z. Indeed, f is a Laplacian eigenfunction in a ball of radius $\asymp \log y \gg 1$ about z.

p. 60 Proposition 5.40.: see below.

p. 60–61 The end of the proof of Proposition 5.41: We cannot just combine those two envelopes. While $\{f - \operatorname{trunc} f : f \in H(\Gamma, s(1-s))\}$ has a L^2-holomorphic envelope in $L^2(Y)$ ($Y = \Gamma\backslash\mathbf{H}$), we cannot conclude that it has a L^2-holomorphic envelope with values in the space of smooth functions W: the dominance principle (5.35) does not apply; we don't have a continuous inclusion $L^2(Y) \to W$.

To fix this, let X be the sum of $L^2(Y)$ and W. The dominance principle does apply with the natural continuous inclusions $C^\infty(Y) \hookrightarrow X$ and $L^2(Y) \hookrightarrow X$ (if X is given a sufficiently weak topology, such as L^2_{loc}), and (5.41) holds with W replaced by X.

p. 61 Corollary 5.42.: Replace W by X. The proof is more delicate now, as we are no longer working with smooth functions. To begin, we need to extend the system (5.39) to X. This is no problem; it suffices to interpret differential operators in the weak sense. The system we obtain on X still has a unique solution for $\sigma > 1$: by elliptic regularity, a solution f must be a true, smooth function of w. It remains to argue that f is of polynomial growth. By definition of X, f is the sum of a smooth polynomial growth function g and a (automatically smooth) L^2 function h. Write the Fourier coefficients[1] of f (2.24) as

$$\widehat{f_n}(y) = a_n W_{0,s-1/2}(4\pi|n|y) + b_n W_{0,s-1/2}(-4\pi|n|y)$$

for $n \neq 0$. We want to show that the b_n are 0. We have $\widehat{f_n} = \widehat{g}_n + \widehat{h}_n$. Because g is of polynomial growth, so is $\widehat{g}_n$. Because h is L^2, it is L^1 and so is $\widehat{h}_n$:

$$\int_0^\infty |\widehat{h}_n(y)| \frac{dy}{y^2} \leqslant \int_0^\infty \int_0^1 |h_n(x+iy)| \frac{dxdy}{y^2} < \infty$$

Thus if $b_n \neq 0$, then $W_{0,s-1/2}(-4\pi|n|y)$ is the sum of a polynomial growth function and a L^2-function. Contradiction.

Finally, we want to show that the a_n are not too large. We show that $|a_n| \ll_\epsilon e^{n\epsilon}$ Indeed, integrating the expression for $\widehat{f_n}$ from ϵ to 2ϵ gives

$$|a_n| \ll_\epsilon \exp(O(n\epsilon)) \int_0^1 \int_\epsilon^{2\epsilon} |f(x+iy)| \frac{dxdy}{y^2}$$

[1] It is unfortunate that we (have to?) resort to the Fourier–Whittaker expansion.

where the second factor is finite because f is the sum of a continuous function and a L^2-function. It follows that f is of rapid decay up to its constant term. But its constant term is a Laplacian eigenfunction, so f is of polynomial growth.

It follows that a solution of the system over X must lie in W, and the uniqueness principle applies.

p. 61 Corollary 5.43.: Replace W by X; the proof does not change.

p. 61 The paragraph before (5.44), and the bottom of the page: "All we know is that they are separately continuous and L^2_{loc}-continuous." This is not true. We have that $E(w,s)$ is L^2_{loc} continuous (because holomorphic) and smooth for fixed s (by elliptic regularity) so that in particular $E(w,s)$ is a true function, but we do not know that it is continuous for fixed w! Similarly as in (5.44) we have:

$$\int_{\mathbf{H}} E(w,s)\partial_s\phi(w,s)dw = \int_{\mathbf{H}} \partial_s E(w,s)\phi(w,s)dw$$

where the derivatives are in the L^2_{loc}-sense (simply by the product rule in L^2_{loc}). It follows that $E(w,s)$ is holomorphic in the distributional sense. We show that $E(w,s)$ is (jointly) measurable. It is continuous (hence measurable) for fixed s. For each $n>0$, divide $\mathbb{C}$ into squares of side length $\frac{1}{n}$ and choose in each square R a point s_R. This defines a sequence $E_n(w,s) = E_n(w,s_{R(s)})$ of measurable pointwise approximations of $E(w,s)$. It converges for fixed s in L^2_{loc} to $E(w,s)$, hence there exists a subsequence which converges almost everywhere to $E(w,s)$. In particular, $E(w,s)$ is measurable. By L^2_{loc}-continuity and Fubini it is jointly L^2_{loc}, and elliptic regularity implies that (after multiplying by a suitable complex polynomial in s) $E(w,s)$ is jointly smooth up to a set of measure 0 in $\mathbf{H}\times U$. (Note here that we don't really need elliptic regularity for overdetermined systems, since $\Delta_{\mathbf{H}} - s(1-s)$ and Δ_U have the same degree: $E(w,s)$ is also annihilated by their sum, which is elliptic.) Call $F(w,s)$ the function that equals $E(w,s)$ up to a set of measure 0 and which is jointly smooth. We want to show that $E=F$ everywhere. We have that $E-F$ is continuous for fixed s. Suppose it is nonzero at some point (w,s). Then some local L^2-norm of $E(\cdot,s)$ is nonzero. By L^2_{loc}-continuity, that local L^2-norm is bounded away from 0 in a neighborhood of s. This contradicts that $E-F$ is zero a.e.

p. 81 Proposition B.22, statement 1.: the last statement is not true: boundedness of the partial derivatives does not imply that f is C^0_b-smooth;

one needs the derivatives to be bounded locally independently of s (e.g. if M is compact). But we don't use this anywhere.